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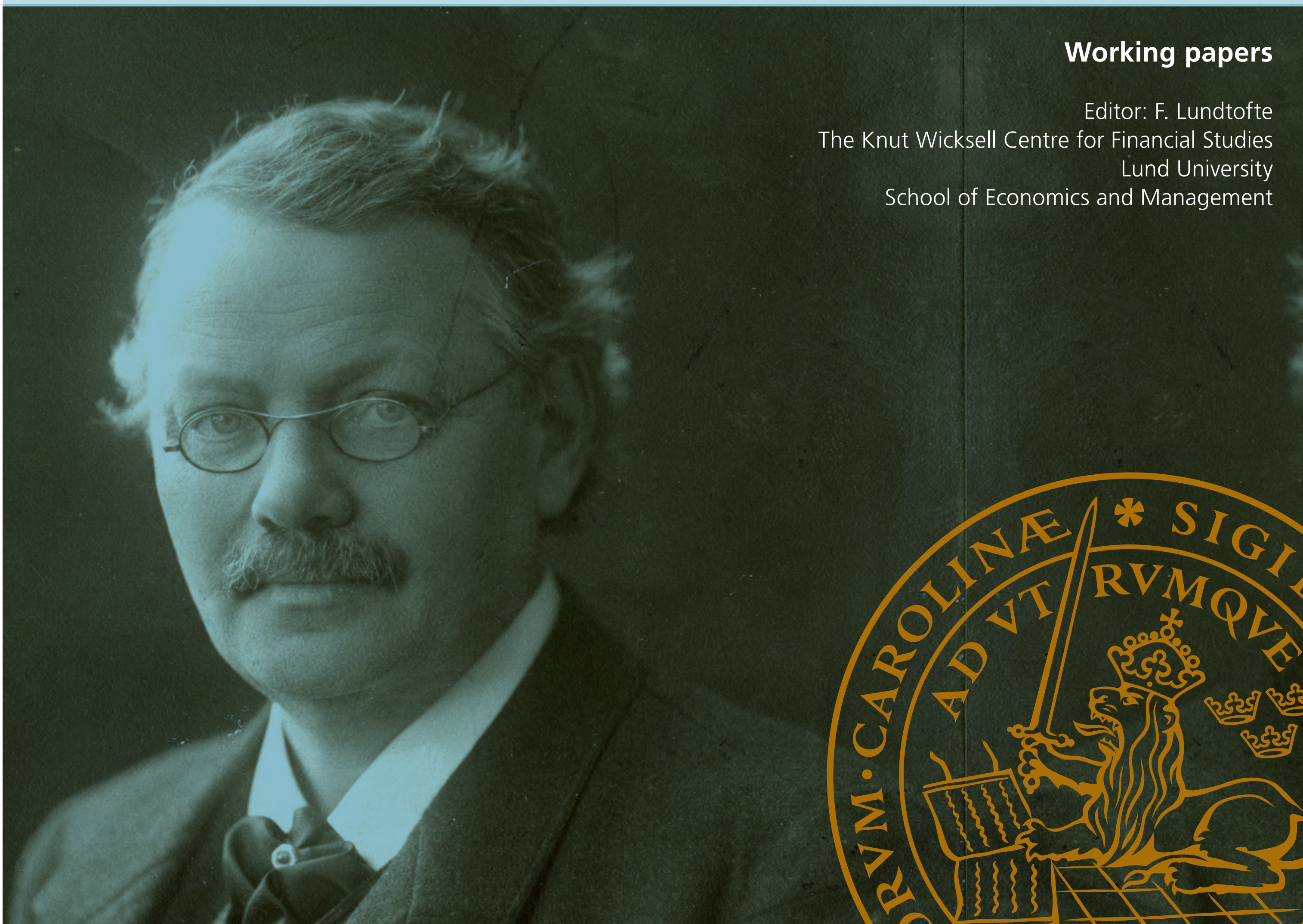

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Closed Form Valuation of Three-Asset Spread Options

With a view towards Clean Dark Spreads

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Abstract

We perform a slight generalization of the Bjerksund and Stensland (2011) spread option valuation formula to cover three-asset spread options. We investigate the pricing performance of the model against the corresponding version of the Kirk formula and the true price calculated with Monte Carlo methods. The numerical setting of the evaluation is designed to mimic a real market situation in the German OTC market for clean dark spread options. The results show that both models give similar and accurate price estimates (compared to the true option price). Comparing the performance between the models we conclude that the three-asset Bjerksund-Stensland formula performs marginally better compared to the three-asset Kirk formula (counting the number of test cases with the lowest absolute pricing error against the true option price).

JEL classification: C6, D81, G12, G13, Q4
Keywords: Clean dark spreads; Energy markets; Financial derivatives; Spread options

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1 Introduction

The European Union Emissions Trading Scheme (EU-ETS) was launched in 2005 with the objective to form an international market place for the price of CO2 emissions. This had a significant impact on the European power generation industry, which comprises a significant proportion of fossil fueled generation assets. As a result, the production cost of electricity in a fossil fueled plant does not only depend on the fuel price, but also on the level of the CO2 price. It is well-known that fossil fueled power plants might be regarded as real-options. Specifically, they take the form of spread options where the price differential is between the outright electricity price and the production cost, where the production cost is a function of the relevant fuel price and the CO2 price. Financially settled spread options (without any linkage to physical plants) are also traded OTC. Such contracts constitute natural hedging tools to manage the market risk of physical power plants.

In order for plant owners and other market participants to regard spread options as a proper hedging alternative accurate and simple valuation models are required. In the general case it is not possible to find a closed form valuation formula for spread options. One therefore need to rely on either time consuming simulation models or closed form approximation formulas. Since practitioners are focused on timely and efficient solutions a closed form approximation formula is typically the preferred choice. To our knowledge the literature on closed form approximations for spread options started with Wilcox (1990), who proposed to model the spread as an arithmetic Brownian motion without any account of the asset specific prices processes and their dependencies. Evidently such an approach does not allow for asset specific greeks. Shimko (1994) developed the model by Wilcox (1990) by adding higher order terms to better approximate the true lognormal solution. Despite the inclusion of the higher order terms the approximation performs quite poorly since it's based on a one-factor model and cannot capture the full dynamics of the true two-factor model. An additional approximation was suggested by Kirk (1995) who use a lognormal specification of the two asset processes adjusting the strike price and replacing the difference of the asset prices by the ratio. It turns out that this approximation is quite accurate under most circumstances, and according to Bjerksund and Stensland (2011) the "Kirk formula" has reached the status of being the "market standard" in many practical applications. In a later paper Carmona and Durrleman (2003) propose approximation formulas for the lower and upper bounds of multi-asset spread options by solving a nonlinear optimization problem. The authors show that their model has a lower tracking error than the Kirk model.[2] The research on multi-commodity spread options continue in Li, Deng and Zhou (2006) and Li, Zhou and Deng (2010) where the authors in the respective papers suggest closed form approximation formulas for two-asset and multi-asset spread options. The formulae are based on a second-order approximation of the exercise boundary and numerical investigations show that they

[2]The tracking error is defined as the the difference between the payoff of the option at maturity and the value of the discretely re-balanced replicating portfolio.

yield highly accurate results. A promising approach to approximate the price of two-asset spread options was given in Bjerksund and Stensland (2011). The authors derive a formula for the spread call price conditional on following the feasible but non-optimal exercise strategy in Kirk (1995). Despite the simplicity of the formula, numerical investigations show that it gives extremely accurate results, indeed outperforming the Kirk formula.

In this paper we specifically treat pricing of three-asset spread options with a view towards clean dark spreads. For the reasons given above three-asset spread options are of particular interest in the context of energy markets. Due to the high accuracy of the two-asset version of the Bjerksund and Stensland (2011) formula, we ask weather a three-asset version of the formula will yield results of similar accuracy. We investigate this by generalizing the original two-asset formula to cover the three-asset case. Similar to the original paper by Bjerksund and Stensland (2011) we investigate the pricing performance of the model against the three-asset version of the Kirk formula (originally proposed in Li et al. (2010) and later examined in Alòs, Eydeland and Laurence (2011)) and the true price estimated with Monte Carlo methods. We conduct the numerical investigation using market reflective parameter values from OTC traded clean dark spread options.

The paper is organized as follows. Section 2 briefly overviews the market for clean dark spreads, which is a specific example of a three-asset spread option. In section 3 we specify the model and derive the pricing formula. Section 4 presents the numerical results. We finally state our conclusions in Section 5.

2 Spread Options in the European Power Markets

Spread options are natural hedging tools for power plants since the plant value explicitly depends on the price difference, i.e. the spread, between the power price and the generation cost of the plant, where the generation cost directly relates to the fuel cost. Due to the environmental impact of thermal fuels the CO2 price also comes into play, and roughly the generation cost is a weighted sum of the fuel price and the CO2 price. Hence, the relevant spread option for power plant hedging has three legs given by (i) the power price, (ii) the fuel price and (iii) the CO2 price.

There exists a semi official European OTC market for spread option contracts. To our knowledge one of the leading players in the German market regularly publish indicative bids and offers for Clean Dark Spreads (CDS) and Clean Spark Spreads (CSS). The quotes are typically plain option single expiry contracts where the underlying product is a standard forward contract with monthly, quarterly or yearly granularity. Other more complicated deal specifications are also quoted. A popular style of contract is the Virtual Power Plant (VPP), which is designed to mimic the cash flows of a real world power pant. Such contracts are specified as a strip of independent options with hourly, daily or monthly exercise. Bilateral negotiations on tailor-made options or VPPs also occur telephonically. The following spread definitions are commonly used in the German power market

$$
\begin{aligned}
CDS(t) &= PowerBase(t) - 0.4Coal(t) - 0.9CO2(t) \qquad (1)\\
CSS(t) &= PowerPeak(t) - 2Gas(t) - 0.4CO2(t) \qquad (2)
\end{aligned}
$$

where $PowerBase/Peak(t)$ denotes the base/peak price of power (in EUR/MWh), and $Coal(t)$ is the price of coal (EUR/mt), and $Gas(t)$ is the gas price (EUR/MWh) and $CO2(t)$ is the carbon price (EUR/mt), where the relevant price references for coal and gas are API2 and TTF. [3] The constants pre-multiplied with the fuel and CO2 components in (1) and (2), respectively relate to the plant efficiency and the number of carbon credits necessary to cover production. We note that the CDS definition in (1) is used throughout this paper.

Three-legged spread options have become increasingly important in the energy commodity markets, especially since the introduction of the CO2 markets. However, standard valuation models for such options are typically based on simulations and hence time consuming. Practitioners obviously prefer closed form solutions. In the next section we slightly generalize the two-asset closed form valuation model by Bjerksund and Stensland (2011) to the three-asset case. We then proceed to investigate its pricing performance against the three-asset version of the famous Kirk model.

3 The Model

We consider a European call option on the three-asset price spread given by $S_1(T) - S_2(T) - S_3(T) - K$, where $K \geq 0$ denote the strike price and T is the expiration date. The terminal payoff of the option can hence be expressed as

$$C(T) = (S_1(T) - S_2(T) - S_3(T) - K)^+ \qquad (3)$$

Standard results suggest that the time 0 price of the call option is given by the risk-neutral expectation of the terminal payoff, discounted with the risk-free interest rate r.

$$C(0) = e^{-rT} E_{\mathbb{Q}}[(S_1(T) - S_2(T) - S_3(T) - K)^+ | \mathfrak{F}_0] \qquad (4)$$

where $\mathbb{Q}$ denote a risk-neutral probability measure. To calculate the expectation in (4) we follow the approach of Bjerksund and Stensland (2011), who employ the implicit strategy given by the Kirk formula, which is to exercise whenever the price of the long asset exceeds a given power function of the prices of the short assets. This is equivalent to an exercise strategy based on the assumption that the sum of the strike and the second and third spread

[3] API2 is a price index calculated by an average of the Argus cif (cost,insurance and freight) ARA (Amsterdam, Rotterdam and Antwerpen) assessment and McCloskeys northwest European steam coal marker and it is the primary price preference for coal contracts in northwest Europe. The Title Transfer Facility (TTF) is a virtual market place for natural gas in the Netherlands. It is a standard price reference for gas deals in the Netherlands and Germany.

legs is lognormal. We assume that the prices of our assets at the future date T can be expressed as

$$S_1(T) = F_1 \exp\left\{-\frac{1}{2}\sigma_1^2 T + \sigma_1 dW_1(T)\right\} \quad (5)$$

$$S_2(T) = F_2 \exp\left\{-\frac{1}{2}\sigma_2^2 T + \sigma_2 dW_2(T)\right\} \quad (6)$$

$$S_3(T) = F_3 \exp\left\{-\frac{1}{2}\sigma_3^2 T + \sigma_3 dW_3(T)\right\} \quad (7)$$

where F_1, F_2, F_3 are prevailing forward prices with delivery at time T, and where W_1, W_2, W_3 are correlated Wiener processes. To clarify the pricing approach we shortly review the two-asset case as given in Bjerksund and Stensland (2011).

In the case where the spread is between two assets it can be verified that the Kirk (1995) formula follows from the expectation

$$C_K = e^{-rT} E_{\mathbb{Q}}\left[\left(S_1(T) - \frac{a(S_2(T))^b}{E_{\mathbb{Q}}[(S_2(T))^b]}\right)^+ |\mathfrak{F}_0\right] \quad (8)$$

where $a = F_2 + K$ and $b = F_2/(F_2 + K)$. Bjerksund and Stensland (2011) use this insight to obtain an alternative spread option approximation formula. They argue that the implicit exercise strategy given by the Kirk formula is to exercise if and only if $S_1(T)$ exceeds a power function of $S_2(T)$. The authors utilize this feasible but non-optimal exercise strategy and express the future payoff of the spread option as

$$C(T) = (S_1(T) - S_2(T) - K) \cdot I\left(S_1(T) \geq \frac{a(S_2(T))^b}{E_{\mathbb{Q}}[(S_2(T))^b]}\right) \quad (9)$$

where I is an indicator function. The option value given by this strategy will represent a lower bound to the true option value. Bjerksund and Stensland (2011) calculate the approximative option value from the payoff in (9), which is in closed form. The authors show that the precision of the approximation is significantly more accurate compared to the Kirk formula.

In this paper we perform a slight generalization of the Bjerksund and Stensland (2011) spread option valuation formula to cover three-asset spread options. The proposed pricing formula is given in the following proposition.

Proposition 3.1 *The approximate value of a three-asset call spread option is given by the formula*

$$\begin{aligned} c(a, b_2, b_3) &= e^{-rT} E_{\mathbb{Q}}\left[(S_1(T) - S_2(T) - S_3(T) - K) \cdot I\left(S_1(T) \geq \frac{a(S_2(T))^{b_2}(S_3(T))^{b_3}}{E_{\mathbb{Q}}[(S_2(T))^{b_2}(S_3(T))^{b_3}]}\right)\right] \\ &= e^{-rT}\left\{F_1 N(d_1) - F_2 N(d_2) - F_3 N(d_3) - K N(d_4)\right\} \end{aligned}$$

where d_1, d_2, d_3 and d_4 are defined by

$$d_1 = \frac{\ln(F_1/a) + \left(\frac{1}{2}\sigma_1^2 + \frac{1}{2}b_2^2\sigma_2^2 + \frac{1}{2}b_3^2\sigma_3^2 - b_2\sigma_1\sigma_2\rho_{1,2} - b_3\sigma_1\sigma_3\rho_{1,3} + b_2b_3\sigma_2\sigma_3\rho_{2,3}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_1/a) + \left(\frac{1}{2}b_2^2\sigma_2^2 + \frac{1}{2}b_3^2\sigma_3^2 - \frac{1}{2}\sigma_1^2 - b_2\sigma_2^2 + b_2b_3\sigma_2\sigma_3\rho_{2,3} - b_3\sigma_2\sigma_3\rho_{2,3} + \sigma_1\sigma_2\rho_{1,2}\right)T}{\sigma\sqrt{T}}$$

$$d_3 = \frac{\ln(F_1/a) + \left(\frac{1}{2}b_2^2\sigma_2^2 + \frac{1}{2}b_3^2\sigma_3^2 - \frac{1}{2}\sigma_1^2 - b_3\sigma_3^2 + b_2b_3\sigma_2\sigma_3\rho_{2,3} - b_2\sigma_2\sigma_3\rho_{2,3} + \sigma_1\sigma_3\rho_{1,3}\right)T}{\sigma\sqrt{T}}$$

$$d_4 = \frac{\ln(F_1/a) + \left(\frac{1}{2}b_2^2\sigma_2^2 + \frac{1}{2}b_3^2\sigma_3^2 - \frac{1}{2}\sigma_1^2 + b_2b_3\sigma_2\sigma_3\rho_{2,3}\right)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{\sigma_1^2 + b_2^2\sigma_2^2 + b_3^2\sigma_3^2 - 2b_2\sigma_1\sigma_2\rho_{1,2} - 2b_3\sigma_1\sigma_3\rho_{1,3} + 2b_2b_3\sigma_2\sigma_3\rho_{2,3}}$$

and where the constants a, b_2, b_3 are given by

$$a = F_2 + F_3 + K$$

$$b_2 = \frac{F_2}{F_2 + F_3 + K}$$

$$b_3 = \frac{F_3}{F_2 + F_3 + K}$$

Proof. See Appendix. ■

The pricing formula in proposition 3.1 is in closed form, which is preferred by practitioners. According to Bjerksund and Stensland (2011) the model in Kirk (1995) is the current market standard for pricing two-asset spread options. A corresponding three-asset generalization of the Kirk formula was proposed in Li et al. (2010) and restated in Alòs et al. (2011). To our knowledge there is no current market standard to price three-asset spread options and we therefore use the Kirk model as a benchmark in this paper. The pricing formula in the tree-asset version of Kirk's model is given by

$$\begin{aligned} c(a, b_2, b_3) &= e^{-rT} E_{\mathbb{Q}}\left[\left(S_1(T) \geq \frac{a(S_2(T))^{b_2}(S_3(T))^{b_3}}{E_{\mathbb{Q}}[(S_2(T))^{b_2}(S_3(T))^{b_3}]}\right)^+\right] \\ &= e^{-rT}\left\{F_1 N(d_{K,1}) - aN(d_{K,2})\right\} \end{aligned} \tag{10}$$

where $N()$ denote the standard normal cumulative probability function and $d_{K,1}$ and $d_{K,2}$

are defined as

$$d_{K,1} = \frac{\ln(F_1/a) + \frac{1}{2}\sigma_K^2 T}{\sigma_K \sqrt{T}} \tag{11}$$

$$d_{K,2} = d_{K,1} - \sigma_K \sqrt{T} \tag{12}$$

$$\sigma_K = \sqrt{\sigma_1^2 + b_2^2\sigma_2^2 + b_3^2\sigma_3^2 - 2b_2\sigma_1\sigma_2\rho_{1,2} - 2b_3\sigma_1\sigma_3\rho_{1,3} + 2b_2b_3\sigma_2\sigma_3\rho_{2,3}} \tag{13}$$

and where the constants a, b_2, b_3 keep their previous definitions.

4 Numerical Results

Here we compare the accuracy of the proposed model (proposition 3.1) against the corresponding version of the Kirk model in given in equations (10)-(13). We evaluate the models by comparing their prices to the true option price, which is estimated by a Monte Carlo procedure with 10^6 draws. We control the simulation error by utilizing our proposed spread formula as control variate.

The numerical setting is based on a real world example taken from the German OTC market for clean dark spreads on August 5th, 2013. We price a single expiry option on the underlying price spread between the calendar year 2014 forward products. The spread is defined in (1). The numerical values of the annualized volatilities of the underlying spread legs are indicative implied mid volatilities published on August 5th 2013 by a leading energy commodities broker. For reasons of confidentiality we cannot disclose the numerical values of the volatilities. Forward closing prices for the given trading date are, $F_{power} = 36.49$, $F_{coal} = 62.63$ and $F_{CO2} = 4.55$, where the power price is from the EEX market, the coal price is the API2 price reference, and the CO2 price is the EUA CO2 price from the EEX exchange. The respective price units are given in subsection 2. The time to maturity is $T = 5/6$ and the annual risk free interest rate is assumed to be $r = 0.02$.

We analyze the pricing performance of the model for different combinations of strike and correlation. The strike $K = 7.35$ corresponds to the ATM level. To ensure usage of realistic correlation coefficients between the spread legs, we estimate the correlation matrix using historical price data between January 2, 2008 - August 5, 2013. The estimates are given in (14). Subsequently, we vary the historical estimates to span different segments of the correlation space, always ensuring positive definiteness of the correlation matrix.

$$\bar{\rho} = \begin{pmatrix} 1 & \rho_{power,coal} & \rho_{power,CO2} \\ \rho_{coal,power} & 1 & \rho_{coal,CO2} \\ \rho_{CO2,power} & \rho_{CO2,coal} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.68 & 0.41 \\ 0.68 & 1 & 0.17 \\ 0.41 & 0.17 & 1 \end{pmatrix} \tag{14}$$

The result from the evaluation is presented in the Tables 1, 2 and 3 in the Appendix. The first row in each cell of the tables reports the option value from the three-asset Kirk's formula.

The second row is the true option price estimated with Monte Carlo methods. The third row is the variance of the simulation result, and the fourth row is from the three-asset Bjerksund-Stensland formula. The shaded cells highlight cases where the Bjerksund-Stensland model outperforms/performs equally well as the Kirk model (in terms of absolute pricing error against the true option price).

In Table 1 we display the result from evaluating the model for different strikes and different values in the power/coal correlation ($\rho_{power,coal}$). The results show that both models give highly similar outcomes. In the majority of the test cases both models give price estimates very close to the true (Monte Carlo) option price. Out of 25 test cases in Table 1 the three-asset Bjerksund-Stensland formula performs better (at four decimal precision) than the Kirk model in 11 cases, and performs equally well in two cases. We further note that the Bjerksund-Stensland formula performs especially well in the OTM cases.

In Table 2 we repeat the previous evaluation, with the difference that we now vary the power/CO2 correlation ($\rho_{power,CO2}$). Again, we note that in most cases both models give similar results close to the true option price. Out of the 25 test cases the three-asset Bjerksund-Stensland formula performs better than the Kirk model in 15 cases, and performs equally well in five cases.

Finally, in Table 3 we carry out the same evaluation by varying the coal/CO2 correlation ($\rho_{coal,CO2}$). As in the previous cases we note that both models give similar prices, close to the true option price. Out of the 25 test cases the three-asset Bjerksund-Stensland formula performs better than the Kirk model in six cases, and performs equally well in 11 cases.

5 Conclusions

In this paper we derive a closed form approximation for the price of a three-asset spread option where the underlying assets are log-normal. The model is a straightforward generalization of the two-asset result in Bjerksund and Stensland (2011). We evaluate the model against a three-asset version of the famous Kirk formula and we check its accuracy against the true option price, which is estimated using a Monte Carlo approach. The numerical setting in the evaluation is designed to mimic a real market situation in the German OTC market for clean dark spread options.

We evaluate the pricing performance of the model for different combinations of strike and correlation. The results indicate that the three-asset Bjerksund-Stensland formula and the corresponding Kirk formula both give similar and accurate price estimates (compared to the true option price). Financially settled spread options are natural hedging tools to manage the market risk of physical power plants. Since practitioners are focused on timely and efficient solutions a closed form approximation formula is typically the preferred choice. We conclude that the both models are good candidates of accurate and fast price approximation formulas suitable for practical application.

Comparing the performance between the two models we conclude that the three-asset

Bjerksund-Stensland formula performs marginally better compared to the three-asset Kirk formula (counting the number of test cases with the lowest absolute pricing error against the true option price).

Appendix

Proof of spread option formula

We denote $m(x, y, z; \rho)$ to be the standard tri-variate density function of the normal distribution, with ρ being the 3×3 correlation matrix

$$\rho = \begin{pmatrix} 1 & \rho_{x,y} & \rho_{x,z} \\ \rho_{x,y} & 1 & \rho_{y,z} \\ \rho_{x,z} & \rho_{y,z} & 1 \end{pmatrix}$$

Lemma 5.1 *The following identity holds for the standard normal tri-variate density, where a, b, c are constants.*

$$\exp\left((ax + by + cz) - \frac{1}{2}\left(a^2 + b^2 + c^2 + 2\rho_{x,y}ab + 2\rho_{x,z}ac + 2\rho_{y,z}cb\right)\right) \times m(x, y, z; \rho)$$

$$= m\left(x - (a + \rho_{x,y}b + \rho_{x,z}c), y - (\rho_{x,y}a + b + \rho_{y,z}c), z - (\rho_{x,z}a + \rho_{y,z}b + c)\right)$$

Given the above identity it holds that

$$E\left[\exp\left((ax + by + cy) - \frac{1}{2}(a^2 + b^2 + c^2 + 2\rho_{x,y}ab + 2\rho_{x,z}ac + 2\rho_{y,z}cb)\right) h(x, y, z)\right]$$

$$= E\left[h\left(x + (a + \rho_{x,y}b + \rho_{x,z}c), y + (\rho_{x,y}a + b + \rho_{y,z}c), z + (\rho_{x,z}a + \rho_{y,z}b + c)\right)\right]$$

for any function $h(x, y, z)$.

To economize on notation we express our system of asset dynamics as

$$X_1 = F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1\epsilon_1\right\}$$

$$X_2 = F_2 \exp\left\{-\frac{1}{2}v_2^2 + v_2\epsilon_2\right\}$$

$$X_3 = F_3 \exp\left\{-\frac{1}{2}v_3^2 + v_3\epsilon_3\right\}$$

With straightforward calculations we can show that

$$\frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]} = a\exp\left\{-\frac{1}{2}(b_2^2v_2^2 + b_3^2v_3^2) - \rho_{2,3}b_2b_3v_2v_3 + b_2v_2\epsilon_2 + b_3v_3\epsilon_3\right\}$$

Given the above results and definitions we proceed to calculate the price of the spread option. From the first part of Proposition 3.1 we know that the price requires an explicit calculation of

$$
\begin{aligned}
& E_{\mathbb{Q}}\left[(X_1 - X_2 - X_3 - K)\cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] \\
= \; & E_{\mathbb{Q}}\left[X_1\cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] - E_{\mathbb{Q}}\left[X_2\cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] \\
- \; & E_{\mathbb{Q}}\left[X_3\cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] - KE_{\mathbb{Q}}\left[\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right]
\end{aligned}
$$

In the following we evaluate each term in the order given above. Note, in the evaluation of the first term we make use Lemma 5.1 in the second equality.

$$
\begin{aligned}
& E_{\mathbb{Q}}\left[X_1\cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] \\
= \; & E_{\mathbb{Q}}\Bigg[F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1\epsilon_1\right\}\cdot I\Bigg(F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1\epsilon_1\right\} \\
& \geq a\exp\left\{-\frac{1}{2}(b_2^2v_2^2 + b_3^2v_3^2) - \rho_{2,3}b_2b_3v_2v_3 + b_2v_2\epsilon_2 + b_3v_3\epsilon_3\right\}\Bigg)|\mathfrak{F}_0\Bigg] \\
= \; & F_1E_{\mathbb{Q}}\Bigg[I\Bigg(F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1(\epsilon_1 + v_1)\right\} \\
& \geq a\exp\left\{-\frac{1}{2}(b_2^2v_2^2 + b_3^2v_3^2) - \rho_{2,3}b_2b_3v_2v_3 + b_2v_2(\epsilon_2 + \rho_{1,2}v_1) + b_3v_3(\epsilon_3 + \rho_{1,3}v_1)\right\}\Bigg)|\mathfrak{F}_0\Bigg] \\
= \; & F_1E_{\mathbb{Q}}\Bigg[I\Bigg(v_1\epsilon_1 - b_2v_2\epsilon_2 - b_3v_3\epsilon_3 \\
& \geq -\ln\left(\frac{F_1}{a}\right) - \frac{1}{2}\left(v_1^2 + b_2^2v_2^2 + b_3^2v_3^2\right) + b_2v_1v_2\rho_{1,2} + b_3v_1v_3\rho_{1,3} - b_2b_3v_2v_3\rho_{2,3}\Bigg)|\mathfrak{F}_0\Bigg] \\
= \; & F_1N\left(\frac{\ln\left(\frac{F_1}{a}\right) + \frac{1}{2}\left(v_1^2 + b_2^2v_2^2 + b_3^2v_3^2\right) - b_2v_1v_2\rho_{1,2} - b_3v_1v_3\rho_{1,3} + b_2b_3v_2v_3\rho_{2,3}}{\sqrt{v_1^2 + b_2^2v_2^2 + b_3^2v_3^2 - 2b_2v_1v_2\rho_{1,2} - 2b_3v_1v_3\rho_{1,3} + 2b_2b_3v_2v_3\rho_{2,3}}}\right) \\
= \; & F_1N(d_1)
\end{aligned}
$$

The second term is evaluated in a similar way.

$$\begin{aligned}
& E_{\mathbb{Q}}\left[X_2 \cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_{\mathbb{Q}}[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] \\
= \quad & F_2 E_{\mathbb{Q}}\Bigg[I\Bigg(F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1(\epsilon_1 + \rho_{1,2}v_2)\right\} \\
& \geq a\exp\left\{-\frac{1}{2}(b_2^2v_2^2 + b_3^2v_3^2) - \rho_{2,3}b_2b_3v_2v_3 + b_2v_2(\epsilon_2 + v_2) + b_3v_3(\epsilon_3 + \rho_{2,3}v_2)\right\}\Bigg)|\mathfrak{F}_0\Bigg] \\
= \quad & F_2 N\left(\frac{\ln\left(\frac{F_1}{a}\right) + \frac{1}{2}\left(b_2^2v_2^2 + b_3^2v_3^2 - v_1^2\right) + b_2b_3v_2v_3\rho_{2,3} - b_2v_2^2 - b_3v_2v_3\rho_{2,3} + v_1v_2\rho_{1,2}}{\sqrt{v_1^2 + b_2^2v_2^2 + b_3^2v_3^2 - 2b_2v_1v_2\rho_{1,2} - 2b_3v_1v_3\rho_{1,3} + 2b_2b_3v_2v_3\rho_{2,3}}}\right) \\
= \quad & F_2 N(d_2)
\end{aligned}$$

The same procedure goes for the third term.

$$\begin{aligned}
& E_{\mathbb{Q}}\left[X_3 \cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_Q[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] \\
= \quad & F_3 E_{\mathbb{Q}}\Bigg[I\Bigg(F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1(\epsilon_1 + \rho_{1,3}v_3)\right\} \\
& \geq a\exp\left\{-\frac{1}{2}(b_2^2v_2^2 + b_3^2v_3^2) - \rho_{2,3}b_2b_3v_2v_3 + b_2v_2(\epsilon_2 + \rho_{2,3}v_3) + b_3v_3(\epsilon_3 + v_3)\right\}\Bigg)|\mathfrak{F}_0\Bigg] \\
= \quad & F_3 N\left(\frac{\ln\left(\frac{F_1}{a}\right) + \frac{1}{2}\left(b_2^2v_2^2 + b_3^2v_3^2 - v_1^2\right) + b_2b_3v_2v_3\rho_{2,3} + v_1v_3\rho_{1,3} - b_2v_2v_3\rho_{2,3} - b_3v_3^2}{\sqrt{v_1^2 + b_2^2v_2^2 + b_3^2v_3^2 - 2b_2v_1v_2\rho_{1,2} - 2b_3v_1v_3\rho_{1,3} + 2b_2b_3v_2v_3\rho_{2,3}}}\right) \\
= \quad & F_3 N(d_3)
\end{aligned}$$

And finally, the evaluation of the fourth term is more simple since we do not have to make use of Lemma 5.1.

$$
\begin{aligned}
& E_{\mathbb{Q}}\left[K \cdot I\left(X_1 \geq \frac{aX_2^{b_2}X_3^{b_3}}{E_Q[X_2^{b_2}X_3^{b_3}]}\right)|\mathfrak{F}_0\right] \\
= \;& KE_{\mathbb{Q}}\left[I\left(F_1 \exp\left\{-\frac{1}{2}v_1^2 + v_1\epsilon_1\right\}\right.\right. \\
& \left.\left.\geq a\exp\left\{-\frac{1}{2}(b_2^2v_2^2 + b_3^2v_3^2) - \rho_{2,3}b_2b_3v_2v_3 + b_2v_2\epsilon_2 + b_3v_3\epsilon_3\right\}\right)|\mathfrak{F}_0\right] \\
= \;& KN\left(\frac{\ln\left(\frac{F_1}{a}\right) + \frac{1}{2}\left(b_2^2v_2^2 + b_3^2v_3^2 - v_1^2\right) + b_2b_3v_2v_3\rho_{2,3}}{\sqrt{v_1^2 + b_2^2v_2^2 + b_3^2v_3^2 - 2b_2v_1v_2\rho_{1,2} - 2b_3v_1v_3\rho_{1,3} + 2b_2b_3v_2v_3\rho_{2,3}}}\right) \\
= \;& KN(d_4)
\end{aligned}
$$

With the following translations $v_1 = \sigma_1\sqrt{T}$, $v_2 = \sigma_2\sqrt{T}$, and $v_3 = \sigma_3\sqrt{T}$ and by defining

$$
\begin{aligned}
& \sqrt{v_1^2 + b_2^2v_2^2 + b_3^2v_3^2 - 2b_2v_1v_2\rho_{1,2} - 2b_3v_1v_3\rho_{1,3} + 2b_2b_3v_2v_3\rho_{2,3}} \\
= \;& \sqrt{(\sigma_1^2 + b_2^2\sigma_2^2 + b_3^2\sigma_3^2 - 2b_2\sigma_1\sigma_2\rho_{1,2} - 2b_3\sigma_1\sigma_3\rho_{1,3} + 2b_2b_3\sigma_2\sigma_3\rho_{2,3})T} \\
= \;& \sigma\sqrt{T}
\end{aligned}
$$

we obtain the result in Proposition 3.1 after risk-free discounting.

Figures and Tables

Spread option prices

	$\rho_{power,coal}$				
K	-0.7	-0.4	0	0.3	0.68**
0	7.3309	7.3100	7.2911	7.2841	7.2821
	7.3324	*7.3115*	*7.2926*	*7.2854*	*7.2826*
	(1.4082e-10)	(1.3620e-10)	(1.2910e-10)	(1.1742e-10)	(8.3361e-11)
	7.3311	7.3103	7.2916	7.2846	7.2821
3	4.5724	4.5019	4.4137	4.3575	4.3129
	4.5773	*4.5068*	*4.4191*	*4.3636*	*4.3192*
	(2.9683e-10)	(3.1167e-10)	(3.4166e-10)	(3.7857e-10)	(4.7554e-10)
	4.5740	4.5035	4.4157	4.3602	4.3158
7.35*	1.5498	1.4121	1.2042	1.0209	0.7252
	1.5546	*1.4173*	*1.2101*	*1.0274*	*0.7321*
	(3.7247e-10)	(3.9890e-10)	(4.2310e-10)	(4.4992e-10)	(4.0175e-10)
	1.5496	1.4119	1.2041	1.0207	0.7245
10	0.5814	0.4744	0.3239	0.2065	0.0634
	0.5816	*0.4752*	*0.3253*	*0.2077*	*0.0621*
	(2.5065e-10)	(2.4881e-10)	(2.3105e-10)	(2.0218e-10)	(1.6612e-10)
	0.5780	0.4716	0.3217	0.2041	0.0588
15	0.0405	0.0226	0.0069	0.0015	0.0000
	0.0389	*0.0218*	*0.0068*	*0.0015*	*0.0000*
	(4.4934e-11)	(3.1446e-11)	(1.2556e-11)	(4.3801e-12)	(2.9803e-13)
	0.0383	0.0214	0.0065	0.0014	0.0000

Table 1: We report spread option values for different strikes (K) and power-coal correlations ($\rho_{power,coal}$). The selected parameter values for the correlation have been chosen in an interval to ensure positive definiteness of the covariance matrix. Remaining correlation parameters ($\rho_{power,CO2}$ and $\rho_{coal,CO2}$) are fixed at the historically estimated levels. The strike level marked with * indicates the ATM-level and the correlation value marked with ** shows the historical estimate. The cases where $K < 7.35$ ($K > 7.35$) are ITM (OTM) options. The first row for each strike reports the option value from the three-asset version of Kirk's formula. The second row (in italics) is the simulation result (10^6 trials). The third row (in parentheses) is the variance of the simulation result. The fourth row is from the three-asset Bjerksund-Stensland formula. Shaded cells highlight cases where the Bjerksund-Stensland model outperforms/perform equally well as the Kirk model (in terms of the absolute pricing error).

Spread option prices

	$\rho_{power,CO2}$				
K	-0.6	-0.3	0	0.41**	0.8
0	7.2979	7.2886	7.2837	7.2821	7.2821
	7.3213	*7.3019*	*7.2893*	*7.2826*	*7.2821*
	(2.6045e-10)	(4.4766e-10)	(4.2752e-10)	(8.2930e-11)	(6.4318e-13)
	7.3174	7.2979	7.2865	7.2821	7.2821
3	4.4505	4.3974	4.3520	4.3129	4.3070
	4.5002	*4.4344*	*4.3756*	*4.3192*	*4.3070*
	(6.1880e-12)	(1.4406e-10)	(4.0462e-10)	(4.8251e-10)	(5.0716e-12)
	4.4998	4.4318	4.3712	4.3158	4.3070
7.35*	1.2980	1.1578	0.9981	0.7252	0.2823
	1.2891	*1.1525*	*0.9971*	*0.7321*	*0.3035*
	(4.0057e-11)	(3.0838e-11)	(6.7749e-11)	(4.0272e-10)	(1.4573e-09)
	1.2863	1.1502	0.9941	0.7245	0.2827
10	0.3899	0.2926	0.1932	0.0634	0.0000
	0.3369	*0.2561*	*0.1733*	*0.0620*	*0.0000*
	(4.5474e-13)	(4.9448e-11)	(1.3859e-10)	(1.6407e-10)	(5.7077e-11)
	0.3369	0.2542	0.1698	0.0588	0.0002
15	0.0125	0.0050	0.0012	0.0000	0.0000
	0.0038	*0.0018*	*0.0005*	*0.0000*	*0.0000*
	(1.6598e-10)	(1.1313e-10)	(3.5711e-11)	(3.2108e-13)	(0.0000)
	0.0010	0.0003	0.0001	0.0000	0.0000

Table 2: We report spread option values for different strikes (K) and power-CO2 correlations ($\rho_{power,CO2}$). The selected parameter values for the correlation have been chosen in an interval to ensure positive definiteness of the covariance matrix. Remaining correlation parameters ($\rho_{power,gas}$ and $\rho_{gas,CO2}$) are fixed at the historically estimated levels. The strike level marked with * indicates the ATM-level and the correlation value marked with ** shows the historical estimate. The cases where $K < 7.35$ ($K > 7.35$) are ITM (OTM) options. The first row for each strike reports the option value from the three-asset version of Kirk's formula. The second row (in italics) is the simulation result (10^6 trials). The third row (in parentheses) is the variance of the simulation result. The fourth row is from the three-asset Bjerksund-Stensland formula. Shaded cells highlight cases where the Bjerksund-Stensland model outperforms/perform equally well as the Kirk model (in terms of the absolute pricing error).

Spread option prices

	$\rho_{coal,CO2}$				
K	-0.3	-0.15	0	0.17**	0.8
0	7.2821	7.2821	7.2821	7.2821	7.2852
	7.2821	*7.2821*	*7.2822*	*7.2826*	*7.2913*
	(2.3122e-10)	(8.0115e-12)	(3.0097e-11)	(8.8971e-11)	(2.2389e-10)
	7.2821	7.2821	7.2821	7.2821	7.2894
3	4.3070	4.3071	4.3083	4.3129	4.3626
	4.3072	*4.3082*	*4.3113*	*4.3192*	*4.3845*
	(4.8812e-11)	(1.7638e-10)	(3.5556e-10)	(4.7390e-10)	(1.6897e-10)
	4.3070	4.3071	4.3090	4.3158	4.3820
7.35*	0.4000	0.5260	0.6273	0.7252	1.0087
	0.4230	*0.5416*	*0.6381*	*0.7321*	*1.0072*
	(1.9041e-09)	(1.1582e-09)	(7.1078e-10)	(4.0272e-10)	(3.4220e-11)
	0.4000	0.5259	0.6270	0.7245	0.0052
10	0.0034	0.0155	0.0352	0.0634	0.1897
	0.0040	*0.0171*	*0.0362*	*0.0621*	*0.1721*
	(8.0882e-11)	(1.2010e-10)	(1.4922e-10)	(8.9688e-10)	(8.7170e-11)
	0.0031	0.0152	0.0335	0.0588	0.1694
15	0.0000	0.0000	0.0000	0.0000	0.0008
	0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0004*
	(0.0000)	(2.7878e-16)	(2.9322e-14)	(3.4512e-13)	(1.9280e-11)
	0.0000	0.0000	0.0000	0.0000	0.0002

Table 3: We report spread option values for different strikes (K) and coal-CO2 correlations ($\rho_{coal,CO2}$). The selected parameter values for the correlation have been chosen in an interval to ensure positive definiteness of the covariance matrix. Remaining correlation parameters ($\rho_{power,coal}$ and $\rho_{power,CO2}$) are fixed at the historically estimated levels. The strike level marked with * indicates the ATM-level and the correlation value marked with ** shows the historical estimate. The cases where $K < 7.35$ ($K > 7.35$) are ITM (OTM) options. The first row for each strike reports the option value from the three-asset version of Kirk's formula. The second row (in italics) is the simulation result (10^6 trials). The third row (in parentheses) is the variance of the simulation result. The fourth row is from the three-asset Bjerksund-Stensland formula. Shaded cells highlight cases where the Bjerksund-Stensland model outperforms/perform equally well as the Kirk model (in terms of the absolute pricing error).

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Closed Form Valuation of Three-Asset Spread Options With a view towards Clean Dark Spreads

RIKARD GREEN

We perform a slight generalization of the Bjerksund and Stensland (2011) spread option valuation formula to cover three-asset spread options. We investigate the pricing performance of the model against the corresponding version of the Kirk formula and the true price calculated with Monte Carlo methods. The numerical setting of the evaluation is designed to mimic a real market situation in the German OTC market for clean dark spread options. The results show that both models give similar and accurate price estimates (compared to the true option price). Comparing the performance between the models we conclude that the three-asset Bjerksund-Stensland formula performs marginally better compared to the three-asset Kirk formula (counting the number of test cases with the lowest absolute pricing error against the true option price).

JEL classication: C6, D81, G12, G13, Q4
Keywords: Clean dark spreads; Energy markets; Financial derivatives; Spread options

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