

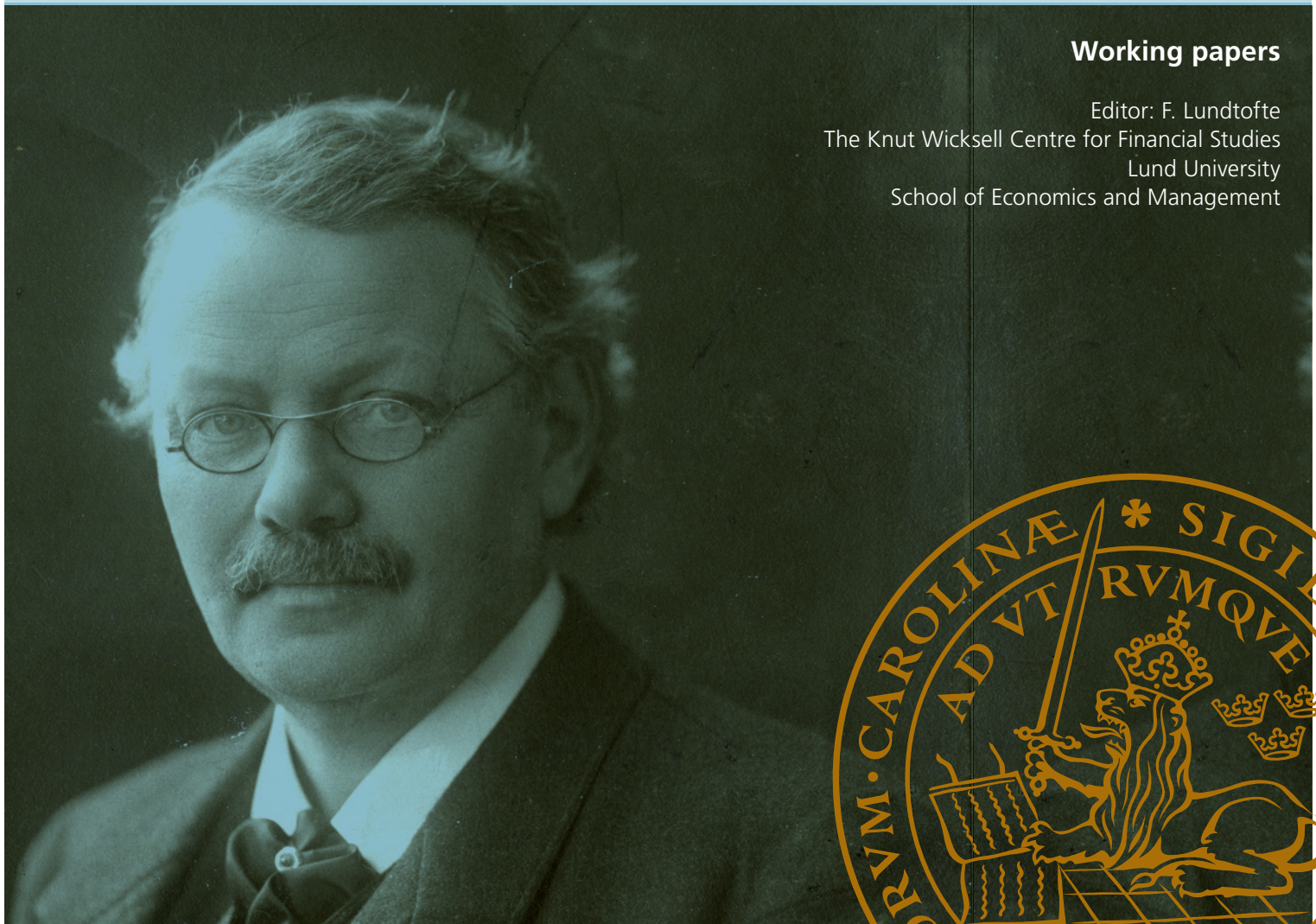
Kelly Trading and Option Pricing

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KELLY TRADING AND OPTION PRICING

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ABSTRACT

In this paper we show that a Kelly trader is indifferent to trade the derivative if and only if the no-arbitrage price is uniquely given by the minimal martingale measure no-arbitrage price, thus providing a natural selection mechanism for option pricing in incomplete markets. We also show that the unique Kelly indifference price results in market equilibrium, in the sense that no Kelly trader can improve the magnitude of his instantaneous Sharpe ratio, by trading the derivative, given the actions of the other market participants.

Keywords Option pricing · Incomplete markets · Hansen-Jagannathan bound · Minimal martingale measure · Kelly Indifference price · Kelly Equilibrium

1 Introduction

In this paper we apply the optimal growth theory of Kelly [12] and Latané [13] to option pricing in incomplete markets. When the market is incomplete absence of arbitrage is not enough to determine a unique price for a derivative. Instead, absence of arbitrage produces a range of possible prices that are all consistent with no arbitrage. Which price to pick is often a matter of taste though some candidates have proven more popular than others. In this paper we give special attention to the arbitrage-free price obtained by using the so-called minimal martingale measure of Föllmer and Schweizer [7]. The range of arbitrage-free prices is in general considered far too wide to be of practical relevance. This led Cochrane and Saá Requejo [5] to define the concept of a no-good-deal price. Similar to no-arbitrage pricing the no-good-deal pricing generates a range of prices all consistent with no good-deals. The price intervals are indexed with a real positive parameter and form an increasing sequence within the no-arbitrage price range. In the limit as the index approaches zero we are left with a unique price given by the minimal martingale measure no-arbitrage price. In the limit as the index approaches infinity the no-good-deal price range coincides with the no-arbitrage price range given additional conditions. The mathematical properties of the optimization problems leading to the upper and lower no-good-deal bounds have formally been characterized in [3]. In short one can view the no-good-deal bounds as an arbitrage-free price with an additional constraint on the instantaneous Sharpe ratio of the derivative. The constraint is, however, enforced via the market price of risk process using the Hansen-Jagannathan bounds derived in [8]. By applying a reasonable bound for the instantaneous Sharpe ratio of the derivative the no-good-deal bound tightens to the extent that it becomes practical relevant.

What makes the optimal growth theory ideal to use in this setting is that these trading strategies have a maximal instantaneous Sharpe ratio. Hence, they always attain the Hansen-Jagannathan bounds. This allows us to quantify the additional gain a trader can make by adding a derivative to his existing portfolio. We call such trading strategies Kelly strategies using the terminology in [2], while other sources, like [16], denote them fractional Kelly strategies. The contribution of this paper is to show that for each no-good-deal price range, identified by the fixed index parameter, a Kelly investor is indifferent to trade the derivative if and only if the price coincide with the minimal martingale measure no-arbitrage price. Hence, this approach is similar in spirit to the utility indifference pricing of Davis [6], although no specific utility specification of the investors is required. We further show that the only price within the (non-exploding)

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no-arbitrage price range that is consistent with a Kelly trader being indifferent to trade the derivative is the minimal martingale measure no-arbitrage price. In order to strengthen our result we also show that this price corresponds to the market being in equilibrium. Hence, similar to the original paper of Black and Scholes [4] we use equilibrium argument to derive a unique price for the derivative. In this regard, we extend their approach to an incomplete market setting.

2 Modeling the Market

We consider a capital market consisting of a bank account B and a number of assets $P = (P_1, \dots, P_N)'$. An asset related to a dividend paying stock is seen as a fund with the dividends re-invested. All assets are assumed to be adapted stochastic processes living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}(t) : t \geq 0\}$ is a right-continuous increasing family of σ -algebras such that $\mathcal{F}(0)$ contains all the \mathbb{P} -null sets of \mathcal{F} . As usual we think of the filtration \mathbb{F} as the carrier of information. We assume that all stochasticity is generated by an M -dimensional standard Brownian motion W , where $M \geq N$, and we identify the filtration \mathbb{F} with the \mathbb{P} -augmentation of the natural filtration of W . Furthermore, we write

$$W(t) = \begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}, \quad (1)$$

where $W^1 = (W_1, \dots, W_N)'$ contains the first N components of W , while $W^2 = (W_{N+1}, \dots, W_M)'$ contains the remaining ones. The reason for this decomposition is that we want to separate the Brownian components driving the stochasticity of the asset prices from those needed to make the model parameters measurable.

We assume that the bank account is locally risk-free with

$$d \log B(t) = r(t) dt, \quad (2)$$

and thus fully determined by an \mathbb{F} -adapted interest rate process r . Regarding the risky assets we set

$$d \log P_n(t) = \mu_n(t) dt + \Sigma'_n(t) dW^1(t), \quad 1 \leq n \leq N, \quad (3)$$

where the logarithmic drift $\mu = (\mu_1, \dots, \mu_N)'$ and the volatility matrix $\Sigma = (\Sigma_1, \dots, \Sigma_N)'$ are arbitrary \mathbb{F} -adapted processes. We note that each volatility vector process Σ_n , $1 \leq n \leq N$, takes values in \mathbb{R}^N and throughout this paper we assume that Σ is a.s. invertible. We also let $\sigma = (\|\Sigma_1\|, \dots, \|\Sigma_N\|)'$ be the vector of real-valued asset volatilities and write $\sigma_{diag} = \text{diag}(\sigma)$ for the associated diagonal matrix. The instantaneous asset-asset covariance matrix V can then be expressed in terms of the corresponding instantaneous correlation matrix ρ according to

$$V(t) = \Sigma(t) \Sigma'(t) = \sigma_{diag}(t) \rho(t) \sigma_{diag}(t). \quad (4)$$

We further introduce the instantaneous Sharpe ratios $s = (s_1, \dots, s_N)'$, as defined in [15], according to

$$s_n(t) = \frac{1}{2} \sigma_n(t) + \frac{\mu_n(t) - r(t)}{\sigma_n(t)}, \quad 1 \leq n \leq N. \quad (5)$$

In order to better understand the separation of the Brownian motion W into the components W^1 and W^2 we introduce a number of non-tradable indices $I = (I_1, \dots, I_{M-N})'$ according to

$$dI(t) = \gamma(t) dt + \Gamma(t) dW^1(t) + \tilde{\Gamma}(t) dW^2(t), \quad (6)$$

where as usual the model parameters of the indices are taken to be arbitrary \mathbb{F} -adapted processes. More specifically, we set $\gamma = (\gamma_1, \dots, \gamma_{M-N})'$, $\Gamma = (\Gamma_1, \dots, \Gamma_{M-N})'$ and $\tilde{\Gamma} = (\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_{M-N})'$, such that the components of Γ take values in \mathbb{R}^N while the components of $\tilde{\Gamma}$ take values in \mathbb{R}^{M-N} . Throughout this paper we allow the indices to represent, say, the interest rate r or the drift of a particular asset μ_n or the volatility σ_n , according to our choice. As will be shown later the purpose of introducing the indices is to be able to work within an extended Markovian state space.

In addition to the tradable assets (B, P) and the non-tradable indices I we introduce one derivative asset whose price process is denoted by Π . This financial contract comes with an expiry date, T , at which time the contract pays the amount

$$\Pi(T) = \Phi(P(T)) \quad a.s., \quad (7)$$

for some deterministic function $\Phi : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$. To be precise a derivative of this form is said to be European and in this paper we solely focus on these types. We do stress, however, that more complex derivatives can often be treated in much a similar way. We now assume that the price of the derivative evolves according to

$$d \log \Pi(t) = \mu_\pi(t) dt + \Sigma'_\pi(t) dW^1(t) + \tilde{\Sigma}'_\pi(t) dW^2(t), \quad 0 \leq t \leq T, \quad (8)$$

for some \mathbb{F} -adapted processes $(\mu_\pi, \Sigma_\pi, \tilde{\Sigma}_\pi)$. Similar to Eq. (5) we also define the instantaneous Sharpe ratio of the derivative

$$s_\pi(t) = \frac{1}{2}\sigma_\pi(t) + \frac{\mu_\pi(t) - r(t)}{\sigma_\pi(t)}, \quad \sigma_\pi^2(t) = \|\Sigma_\pi(t)\|^2 + \|\tilde{\Sigma}_\pi(t)\|^2. \quad (9)$$

Finally, we let $\rho_\pi = (\rho_{\pi,1}, \dots, \rho_{\pi,N})'$ denote the instantaneous correlation between the derivative and each of the risky assets according to

$$\rho_{\pi,n}(t) = \frac{\Sigma'_\pi(t) \Sigma_n(t)}{\sigma_\pi(t) \sigma_n(t)}, \quad 1 \leq n \leq N. \quad (10)$$

With the notation for the tradable assets and the non-tradable indices put in place we now turn our attention to trading. An investor can invest in the tradable assets and for the sake of simplicity we assume that there are no transaction fees, that short-selling is allowed, that trading takes place continuously in time, and that the investor's trading activity does not impact the asset prices. We define a trading strategy as an \mathbb{F} -predictable process $q = (q_0, q_1, \dots, q_{N+1})'$ representing the number of shares held in each asset such that the corresponding portfolio process takes the form

$$X(t) = q_0(t) B(t) + \sum_{n=1}^N q_n(t) P_n(t) + q_{N+1}(t) \Pi(t). \quad (11)$$

In order to analyze the performance of the portfolio process we impose the restriction that, when re-balancing the portfolio, money can neither be injected nor withdrawn. Such trading strategies are said to be self-financing and imply that the portfolio process evolves according to

$$X(t) = X(0) + \int_0^t q_0(s) dB(s) + \sum_{n=1}^N \int_0^t q_n(s) dP_n(s) + \int_0^t q_{N+1}(s) d\Pi(s). \quad (12)$$

In many applications it is often more convenient to define the self-financing trading strategy as being proportional to the wealth X . For this reason we introduce $w = (w_1, \dots, w_N)'$ and w_{N+1} according to

$$w_n(t) X(t) = q_n(t) P_n(t), \quad w_{N+1}(t) X(t) = q_{N+1}(t) \Pi(t), \quad 1 \leq n \leq N, \quad (13)$$

such that, when expressed in differential form, the portfolio dynamics takes the form

$$\frac{dX(t)}{X(t)} = \left(1 - \sum_{n=1}^{N+1} w_n(t)\right) \frac{dB(t)}{B(t)} + \sum_{n=1}^N w_n(t) \frac{dP_n(t)}{P_n(t)} + w_{N+1}(t) \frac{d\Pi(t)}{\Pi(t)}. \quad (14)$$

Straightforward calculations, using the quadratic covariation process, allows us to compute the volatility process

$$\sigma_X^2(t) = \frac{1}{X^2(t)} \frac{d}{dt} [X, X](t) = \frac{d}{dt} [\log X, \log X](t), \quad (15)$$

from which we derive the evolution of the logarithmic portfolio process

$$d \log X(t) = \mu_X(t) dt + (w'(t) \Sigma(t) + w_{N+1}(t) \Sigma'_\pi(t)) dW^1(t) + w_{N+1}(t) \tilde{\Sigma}'_\pi(t) dW^2(t), \quad (16)$$

where

$$\mu_X(t) = r(t) + w'(t) \sigma_{diag}(t) s(t) + w_{N+1}(t) \sigma_\pi(t) s_\pi(t) - \frac{1}{2} \sigma_X^2(t), \quad (17)$$

$$\sigma_X^2(t) = w'(t) V(t) w(t) + w_{N+1}^2(t) \sigma_\pi^2(t) + 2w_{N+1}(t) \sigma_\pi(t) w'(t) \sigma_{diag}(t) \rho_\pi(t). \quad (18)$$

This allows us to compute the instantaneous Sharpe ratio of the portfolio according to

$$s_X(t) = \frac{1}{2} \sigma_X(t) + \frac{\mu_X(t) - r(t)}{\sigma_X(t)}. \quad (19)$$

Of course, in order for the instantaneous Sharpe ratio of the portfolio to be well defined we must impose integrability conditions on the model parameters and on the trading strategy. We also stress that an investor can only engage in trading as long as his wealth is positive. This motivates the definition below.

Definition 2.1. Given a trading horizon T and initial capital $X(0) \geq 0$. A self-financing trading strategy is said to be *admissible* if $X(t) \geq 0$, $0 \leq t \leq T$, a.s. and

$$\mathbb{P} \left(\int_0^T (|\mu_X(t) - r(t)| + \sigma_X^2(t)) dt < \infty \right) = 1.$$

However, this is not enough to ensure the instantaneous Sharpe ratio of the portfolio to be well defined. If, say, we can find a trading strategy such that the volatility $\sigma_X = 0$ a.s., over some time interval, the instantaneous Sharpe ratio might still explode. For this reason we must impose the additional requirement of no-arbitrage in the capital market and in the next section we briefly recap the essential details.

3 No-Arbitrage Pricing

Absence of arbitrage is a very simple and intuitive concept yet its uses in continuous time finance is at times rather technical. The purpose of this section is to summarize the established theory while keeping the technicalities at a minimum. In order to do so we have chosen to work with admissible trading strategies that ensure the portfolio process to be positive. This implies that bankruptcy is an absorbing state, see Remark 3.3.4 in [10], which makes a lot of sense from an economic point of view. However, much of the no-arbitrage results can be applied to a weaker notion of admissible trading strategies. For instance, it is often sufficient to assume that the portfolio process is a.s. bounded from below by some real constant. The problem with such extensions though lie in the practical justifications.

An arbitrage opportunity is a trading strategy for which invested money cannot be lost but profits can be made. Typically such strategies evolve shortening one asset and buying another. Another way to express an arbitrage opportunity is to relate it to the locally risk-free bank account and this is the path we choose.

Definition 3.1. Given a trading horizon T and initial capital $X(0) \geq 0$. An admissible trading strategy is said to be an *arbitrage opportunity* if

$$\mathbb{P}\left(\frac{X(T)}{X(0)} \geq \frac{B(T)}{B(0)}\right) = 1, \quad \mathbb{P}\left(\frac{X(T)}{X(0)} > \frac{B(T)}{B(0)}\right) > 0.$$

A capital market where no arbitrage opportunities exists is said to be *arbitrage-free*.

In order to state what it takes to rule out arbitrage opportunities and prepare the ground for future applications it is beneficial to introduce a number of additional concepts.

Definition 3.2. We say that an \mathbb{R}^M -valued \mathbb{F} -adapted process θ belongs to the space \mathcal{I}_T if

$$\mathbb{P}\left(\int_0^T \|\theta(t)\|^2 dt < \infty\right) = 1.$$

For any $\mathbb{R}^{K \times M}$ -valued \mathbb{F} -adapted process A , where K is an arbitrary integer, we further define the subsets

$$\begin{aligned} \mathcal{K}_T(A) &= \{\theta \in \mathcal{I}_T : \theta(t) \in \text{kernel} A(t), \quad \forall t \in [0, T] \quad a.s.\}, \\ \mathcal{K}_T^\perp(A) &= \{\theta \in \mathcal{I}_T : \theta(t) \in \text{range} A'(t), \quad \forall t \in [0, T] \quad a.s.\}. \end{aligned}$$

The next result shows that the subsets $\mathcal{K}_T(A)$ and $\mathcal{K}_T^\perp(A)$ are orthogonal and that the decomposition can be computed given the pseudo-inverse A^+ of A .

Lemma 3.3. Given an $\mathbb{R}^{N \times M}$ -valued \mathbb{F} -adapted process A . Every process $\theta \in \mathcal{I}_T$ admits a unique orthogonal decomposition

$$\theta(t) = \theta_{\parallel}(t) + \theta_{\perp}(t), \quad \theta_{\parallel} \in \mathcal{K}_T(A), \quad \theta_{\perp} \in \mathcal{K}_T^\perp(A).$$

In terms of the pseudo-inverse A^+ of A the decomposition takes the form

$$\theta_{\parallel}(t) = (I - A^+(t)A(t))\theta(t), \quad \theta_{\perp}(t) = A^+(t)A(t)\theta(t).$$

Proof. We first note that the decomposition is valid since $\theta_{\parallel} + \theta_{\perp} = \theta$. Next, from the definition of a pseudo-inverse we know that $AA^+A = A$ and $(A^+A)^\perp = A^+A$. This shows that θ_{\parallel} is in the kernel of A and that

$$\theta_{\perp}'(t)\theta_{\parallel}(t) = \theta'(t)A^+(t)A(t)(I - A^+(t)A(t))\theta(t) = 0.$$

Hence, the decomposition is orthogonal. Finally, since the pseudo-inverse further satisfies $A^+ = A'(AA')^+$ it follows that θ_{\perp} is in the range of A' . For additional details on pseudo-inverses see [1]. \square

We now apply these results to the capital market (B, P, Π) consisting of the bank account, the risky assets and the derivative. The results below are taken from [10] where additional information is provided.

Proposition 3.4. In an arbitrage-free capital market there exists a process $\theta \in \mathcal{I}_T$, known as the market price of risk, such that

$$\begin{pmatrix} \sigma_{diag}(t) s(t) \\ \sigma_{\pi}(t) s_{\pi}(t) \end{pmatrix} = \begin{pmatrix} \Sigma(t) & \mathbf{0}_{N \times M-N} \\ \Sigma'_{\pi}(t) & \Sigma'_{\pi}(t) \end{pmatrix} \theta(t).$$

We let Θ_T denote the collection of such processes $\theta \in \mathcal{I}_T$.

Proof. Let us introduce the notation

$$b = \begin{pmatrix} \sigma_{diag}(t) s(t) \\ \sigma_\pi(t) s_\pi(t) \end{pmatrix}, \quad A = \begin{pmatrix} \Sigma(t) & \mathbf{0}_{N \times M-N} \\ \tilde{\Sigma}'_\pi(t) & \tilde{\Sigma}'_\pi(t) \end{pmatrix},$$

such that we can write the equation to be proved as $b = A\theta$. We want to show that absence of arbitrage opportunities implies that b is in the range of A , that is $b \in \mathcal{K}_T^\perp(A')$, but this is equivalent to show that there exist arbitrage opportunities if $b \in \mathcal{K}_T(A')$. Now consider the self-financing trading strategy defined by

$$\hat{w}(t) = \begin{pmatrix} w(t) \\ w_{N+1}(t) \end{pmatrix} = \frac{k}{\|b(t)\|^2} b(t), \quad k > 0.$$

Hence, for $b \in \mathcal{K}_T(A')$ it follows that $\hat{w} \in \mathcal{K}_T(A')$. Therefore, according to Eq. (16), the portfolio process satisfies

$$d \log X(t) = (r(t) + k) dt \quad \Leftrightarrow \quad \frac{X(T)}{X(0)} = \frac{B(T)}{B(0)} e^{kT}.$$

This shows that \hat{w} is an arbitrage opportunity, according to Definition 3.1, which concludes the proof. \square

While the instantaneous Sharpe ratio processes are uniquely defined for each security the market price of risk process is in general not. We illustrate this point using the decomposition $\theta^1 = (\theta_1, \dots, \theta_N)'$ and $\theta^2 = (\theta_{N+1}, \dots, \theta_M)'$.

Theorem 3.5. *Let the capital market be arbitrage-free. Then any market price of risk process*

$$\theta(t) = \begin{pmatrix} \theta^1(t) \\ \theta^2(t) \end{pmatrix} \in \Theta_T,$$

satisfies

$$\begin{aligned} \theta^1(t) &= \Sigma^{-1}(t) \sigma_{diag}(t) s(t), \\ \theta^2(t) &= \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) (\sigma_\pi(t) s_\pi(t) - \Sigma'_\pi(t) \theta^1(t)) + \left(I - \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) \tilde{\Sigma}'_\pi(t) \right) \phi(t), \end{aligned}$$

for some arbitrary \mathbb{F} -adapted process ϕ taking values in \mathbb{R}^{M-N} .

Proof. The solution for θ^1 follows directly, while the expression for θ^2 is constructed with the aid of Lemma 3.3. Straightforward calculations verify that

$$\sigma_\pi(t) s_\pi(t) = \Sigma'_\pi(t) \theta^1(t) + \tilde{\Sigma}'_\pi(t) \theta^2(t).$$

which completes the proof. \square

The sole existence of a market price of risk process $\theta \in \Theta_T$ is enough to rule out a multitude of arbitrage opportunities. Unfortunately, though, it does not rule out all possible arbitrage opportunities. The reason is that a market price of risk process $\theta \in \Theta_T$ can, in a mathematical sense, be too wild, leading to what can be considered as pathological results. For this reason we introduce the additional notation.

Definition 3.6. For $\theta \in \Theta_T$, we introduce the local martingale

$$Z(t) = \exp \left(-\frac{1}{2} \int_0^t \|\theta(s)\|^2 ds - \int_0^t \theta'(s) dW(s) \right), \quad 0 \leq t \leq T. \quad (20)$$

We further let $\Theta_T^M = \{\theta \in \Theta_T : \mathbb{E}[Z(T)] = 1\}$ denote the subclass of processes for which Z is a martingale.

The process Z/B is often referred to as the stochastic discount factor and consequently $\|\theta\|$ represents the associate volatility. The importance of the subspace Θ_T^M was first highlighted in [9] as explained below.

Proposition 3.7. *There exists no arbitrage opportunities if the market price of risk process $\theta \in \Theta_T^M$.*

Proof. While credit is due to [9] we also refer to [10], Theorem 1.4.2, for a comprehensive proof. \square

In order for θ to belong to the subspace Θ_T^M the norm $\|\theta\|^2 = \|\theta^1\|^2 + \|\theta^2\|^2$ must satisfy additional integrability conditions. Leaving aside the technicalities we notice, using Theorem 3.5, that the norm of θ^2 is given by

$$\|\theta^2(t)\|^2 = \|\tilde{\Sigma}_\pi(t)\|^{-2} (\sigma_\pi(t) s_\pi(t) - \Sigma'_\pi(t) \theta^1(t))^2 + \phi'(t) \left(I - \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) \tilde{\Sigma}'_\pi(t) \right) \phi(t). \quad (21)$$

Hence, this norm is minimal when the arbitrary process ϕ equals the zero-vector. In this case we introduce an upper bound on the instantaneous Sharpe ratio of the derivative according to

$$\hat{c} = \sup\{c \geq 0 : \theta \in \Theta_T^M, \theta^1(t) = \Sigma^{-1}(t) \sigma_{diag}(t) s(t), \theta^2(t) = \|\tilde{\Sigma}_\pi(t)\|^{-1} \tilde{\Sigma}_\pi(t) c, 0 \leq t \leq T\} \quad (22)$$

From here onward we assume that the market price of risk process $\theta \in \Theta_T^M$, thereby ensuring the capital market to be arbitrage-free. For each choice of θ we then define a new probability measure

$$\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta(T) \mathbf{1}\{A\}], \quad A \in \mathcal{F}(T), \quad (23)$$

equivalent to \mathbb{P} , such that

$$V_\theta(t) = W(t) + \int_0^t \theta(s) ds, \quad 0 \leq t \leq T, \quad (24)$$

is a \mathbb{P}_θ -Brownian motion according to the Girsanov theorem. A direct application of Itô's lemma now yields

$$d \frac{\Pi(t)}{B(t)} = \frac{\Pi(t)}{B(t)} \left(\Sigma'_\pi(t) dV_\theta^1(t) + \tilde{\Sigma}'_\pi(t) dV_\theta^2(t) \right), \quad (25)$$

where V_θ^1 and V_θ^2 are defined analogously to W^1 and W^2 . Hence, with respect to the \mathbb{P}_θ -measure, the process Π/B is a non-negative local martingale and thus a supermartingale

$$\mathbb{E}_\theta \left[\frac{\Pi(T)}{B(T)} | \mathcal{F}(t) \right] \leq \frac{\Pi(t)}{B(t)}.$$

If we further assume that the derivative payoff, as expressed in Eq. (7), is sufficiently integrable for Π/B to be a martingale and that the additional indices are chosen in such a way as to generate a Markovian system, we may define the arbitrage-free price as below.

Definition 3.8. The no-arbitrage price of the derivative is defined by

$$\frac{\Pi(t)}{B(t)} = \mathbb{E}_\theta \left[\frac{\Phi(P(T))}{B(T)} | P(t), I(t) \right], \quad \theta \in \Theta_T^M, \quad (26)$$

where the expectation is to be computed subject to the dynamics

$$d \log P_n(t) = (r(t) - \frac{1}{2} \sigma_n^2(t)) dt + \Sigma'_n(t) dV_\theta^1(t), \quad 1 \leq n \leq N, \quad (27)$$

$$dI(t) = (\gamma(t) - \Gamma(t) \theta^1(t) - \tilde{\Gamma}(t) \theta^2(t)) dt + \Gamma(t) dV_\theta^1(t) + \tilde{\Gamma}(t) dV_\theta^2(t). \quad (28)$$

By varying θ , or more precisely θ^2 , we see that there exists an interval of prices that are all consistent with no arbitrage. We further observe that only if $\|\tilde{\Gamma}_i\| = 0$, for all i , can we categorically claim that the arbitrage-free price of the derivative is unique. The reason is that, in this case, the terminal payoff can be perfectly replicated by trading in the primary assets as originally proved in [14]. In the general case where $\|\tilde{\Sigma}_\pi\| > 0$, referred to as an incomplete market, a unique price can sometimes be derived given assumptions on the specific structure of the component θ^2 . However, such assumptions generally lead to the price of the derivative approaching the upper no-arbitrage pricing bound, see [11] for details. It is fair to say that the no-arbitrage pricing interval is too wide to be of any practical relevance and that derivative prices observed in the market are typically not in line with the upper no-arbitrage pricing bound. In order to restrict the no-arbitrage pricing interval, in search for market alienated derivative prices, we may introduce a bound on the instantaneous Sharpe ratio of the derivative. This gives rise to a pricing methodology known as no-good-deal pricing, which brings us to the next section.

4 No-Good-Deal Pricing

The concept of no-good-deal pricing was first introduced in [5]. The arguments is based on the historical observation that Sharpe ratios, for most assets, typically fluctuates around the value 0.5. Some assets have a higher ratio, some have a lower, so a fund manager combining assets cannot expect to generate a portfolio with arbitrarily high Sharpe ratio. In fact a portfolio having a Sharpe ratio above one is considered very good, while a Sharpe ratio above two is considered

outstanding. To put these number in perspective we recall a result in [2] showing that the instantaneous Sharpe ratio of any portfolio must be finite if the capital market is arbitrage-free. Hence, there is a huge difference, in numerical value of the Sharpe ratio, between a good deal and an arbitrage opportunity. The no-good-deal pricing can be thought of as imposing a bound on the instantaneous Sharpe ratio of the derivative. However, for technical reasons it turns out to be much easier to impose a bound directly on the market price of risk component θ^2 . The connection between the market price of risk process (or equally the volatility of the stochastic discount factor) and the instantaneous Sharpe ratio of the derivative is known as the Hansen-Jagannathan bound and was first derived in [8]. In our setting their result takes the form

Proposition 4.1. *The instantaneous Sharpe ratio of the derivative is bounded from above by the term*

$$s_\pi^2(t) \leq s'(t) \rho^{-1}(t) s(t) + \|\tilde{\Sigma}_\pi(t)\|^{-2} (\sigma_\pi(t) s_\pi(t) - \Sigma'_\pi(t) \theta^1(t))^2.$$

Proof. Since, $\sigma_\pi s_\pi = \Sigma'_\pi \theta^1 + \tilde{\Sigma}'_\pi \theta^2$ and $\sigma_\pi^2(t) = \|\Sigma_\pi(t)\|^2 + \|\tilde{\Sigma}_\pi(t)\|^2$, it follows from the Cauchy-Schwartz inequality that

$$s_\pi^2(t) \leq \|\theta^1(t)\|^2 + \|\theta^2(t)\|^2.$$

Explicit calculations, using Theorem 3.5, now yield

$$\begin{aligned} \|\theta^1(t)\|^2 &= s'(t) \rho^{-1}(t) s(t), \\ \|\theta^2(t)\|^2 &= \|\tilde{\Sigma}_\pi(t)\|^{-2} (\sigma_\pi(t) s_\pi(t) - \Sigma'_\pi(t) \theta^1(t))^2 + \phi'(t) \left(I - \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) \tilde{\Sigma}'_\pi(t) \right) \phi(t). \end{aligned}$$

Since the Hansen-Jagannathan bound must hold for any vector process ϕ it particularly holds for the zero-vector. \square

We see that the upper bound of the instantaneous Sharpe ratio of the derivative is related to the minimum norm of the market price of risk component θ^2 . We now define the no-good-deal price accordingly.

Definition 4.2. The no-good deal price of the derivative is defined by

$$\frac{\Pi_c(t)}{B(t)} = \mathbb{E}_\theta \left[\frac{\Phi(P(T))}{B(T)} \mid P(t), I(t) \right], \quad \theta \in \Theta_T^M, \quad c \in \mathbb{R}_+,$$

where the expectations is to be computed subject to the dynamics

$$\begin{aligned} d \log P_n(t) &= (r(t) - \frac{1}{2} \sigma_n^2(t)) dt + \Sigma'_n(t) dV_\theta^1(t), \quad 1 \leq n \leq N, \\ dI(t) &= (\gamma(t) - \Gamma(t) \theta^1(t) - \tilde{\Gamma}(t) \theta^2(t)) dt + \Gamma(t) dV_\theta^1(t) + \tilde{\Gamma}(t) dV_\theta^2(t). \end{aligned}$$

Although the definition of the no-good-deal price looks fairly innocent it is far so. The reason is that θ^2 , as expressed in Theorem 3.5, contains information about the derivative price process. Hence, we are faced with some kind of recursive definition. Nevertheless, as shown in [8] and in particular [3], this definition leads to a non-standard Hamilton–Jacobi–Bellman equation which allows us to compute the upper and lower no-good-deal bounds. Note that the no-good-deal price interval is increasing with respect to c and that each price range is a subset of the no-arbitrage price interval.

Proposition 4.3. *The bounds of the no-good-deal price Π_c , $c \geq 0$, correspond to*

$$\theta^2(t) = \pm \|\tilde{\Sigma}_\pi(t)\|^{-1} \tilde{\Sigma}_\pi(t) c. \quad (29)$$

Proof. The proof follows from [3] and is further explained in the Appendix. \square

There are two important conclusions to be drawn from this result. First, we see from Eq. (22) that the no-arbitrage bounds can be obtained from the no-good-deal bounds when $c = \hat{c}$ and secondly, we see that there is still an intricate recursive relationship between Π_c and θ^2 , through the volatility component $\tilde{\Sigma}_\pi$. Leaving aside the details about the explicit calculations we observe that only when $c = 0$, in which case θ^2 equals the zero-vector, can we compute the no-good-deal price in a straightforward way. In this special case the no-good-deal price is unique and corresponds to, what we call, the minimal martingale measure no-arbitrage price. The minimal martingale measure was originally introduced in [7] in a slightly different context. However, it is custom to refer to the martingale measure obtained by letting θ^2 equal the zero-vector as the minimal martingale measure. Note that in the degenerate case where the market is complete, that is $\|\tilde{\Sigma}_\pi\| = 0$, the minimal martingale measure no-arbitrage price equals the unique price of the derivative that allows for a perfect replication of the derivative payoff.

It is interesting to note that had we defined the no-good-deal price based on a constraint on the instantaneous Sharpe ratio we would have had little hope in computing the corresponding upper and lower bounds of the derivative. Hence, the fact that we impose a bound on the market price of risk process rather than on the instantaneous Sharpe ratio is more of a technical nature. After all, it is the instantaneous Sharpe ratio of the derivative that we want to control and in the current setting we see that $s_\pi^2 \leq s' \rho^{-1} s + c^2$. In order to better understand the nature of this bound we generalize Proposition 4.2 to arbitrary portfolios.

Corollary 4.4. *A portfolio that can only trade in the primary assets has an instantaneous Sharpe ratio that satisfies*

$$s_X^2(t) \leq s'(t) \rho^{-1}(t) s(t),$$

while a portfolio that can trade in the primary assets and the derivative satisfies

$$s_X^2(t) \leq s'(t) \rho^{-1}(t) s(t) + \|\tilde{\Sigma}_\pi(t)\|^{-2} (\sigma_\pi(t) s_\pi(t) - \Sigma'_\pi(t) \theta^1(t))^2.$$

Proof. We illustrate the proof by considering the case where the derivative is excluded from the portfolio, that is when $w_{N+1} = 0$. In this case the instantaneous Sharpe ratio of the portfolio equals

$$s_X(t) = \frac{w'(t) \sigma_{diag}(t) s(t)}{\sqrt{w'(t) V(t) w(t)}} = \frac{w'(t) \Sigma(t) \theta^1(t)}{\sqrt{w'(t) V(t) w(t)}}.$$

Hence, by applying the Cauchy-Schwartz inequality we obtain

$$s_X^2(t) \leq \frac{\|\Sigma'(t) w(t)\|^2 \|\theta^1(t)\|^2}{w'(t) V(t) w(t)} = \|\theta^1(t)\|^2.$$

The first result now follows from the proof of Proposition 4.1. The second result is computed analogously and thus omitted. \square

Note that if we can find trading strategies such that the upper bounds of the instantaneous Sharpe ratios are attainable then we can quantify the precise contribution of adding the derivative to the opportunity set. This is the topic of the next section.

5 Kelly Indifference Pricing

In this section we study the investment strategies of a Kelly trader. Following [2] we recall that the Kelly strategy is defined as to maximize the instantaneous Sharpe ratio of a portfolio. We show that a Kelly trader can never reduce the instantaneous Sharpe ratio of his portfolio by adding the derivative. Moreover, if the instantaneous Sharpe ratio of the derivative meets a precise condition the Kelly trader is indifferent whether to trade the derivative. This induces a unique price for the derivative similar in spirit to the utility indifference pricing in [6].

Following [2] we recall that the Kelly strategy maximizes the instantaneous Sharpe ratio of the portfolio. The corresponding drift and volatility of such a trading strategy equal

$$\mu_X(t) = r(t) + \frac{1}{2} k(t) (2 - k(t)) s_X^2(t), \quad (30)$$

$$\sigma_X^2(t) = k^2(t) s_X^2(t), \quad (31)$$

where k is some real-valued \mathbb{F} -adapted process known as the Kelly multiplier. The Kelly strategy is said to be efficient if k takes values in $[0, 1]$ and optimal if $k = 1$. Here optimal refers to the observation that this choice makes the drift of the portfolio maximal. In order to compute the instantaneous Sharpe ratio of an arbitrary portfolio we first present a supplementary technical result.

Lemma 5.1. *Suppose that the matrix*

$$\hat{\rho}(t) = \begin{pmatrix} \rho(t) & \rho_\pi(t) \\ \rho'_\pi(t) & 1 \end{pmatrix},$$

is a.s. positive definite. Then

$$\hat{\rho}^{-1}(t) = \begin{pmatrix} \rho^{-1}(t) + h^{-1}(t) \rho^{-1}(t) \rho_\pi(t) \rho'_\pi(t) \rho^{-1}(t) & -h^{-1}(t) \rho^{-1}(t) \rho_\pi(t) \\ -h^{-1}(t) \rho'_\pi(t) \rho^{-1}(t) & h^{-1}(t) \end{pmatrix},$$

where the real-valued \mathbb{F} -adapted process $h = 1 - \rho'_\pi \rho^{-1} \rho_\pi \in (0, 1)$ a.s.

Proof. Since $\hat{\rho}$ is a.s. positive definite the inverse $\hat{\rho}^{-1}$ exists and is also a.s. positive definite. Furthermore, ρ is a.s. positive definite since every principal submatrix of a positive definite matrix is positive definite. This implies that ρ^{-1} is also a.s. positive definite from which we conclude that $1 - h > 0$. By using the rules for calculating the determinant of a block matrix we further note that

$$\det \hat{\rho}^{-1}(t) = h^{-1}(t) \det \rho^{-1}(t).$$

Finally, since the determinant of a positive definite matrix is strictly positive it follows that $h > 0$. The particular form of $\hat{\rho}^{-1}$ is easily verified by a direct calculation. \square

We are now ready to present the details about the Kelly strategy for various portfolios.

Proposition 5.2. *Let k be a real-valued \mathbb{F} -adapted process. The instantaneous Sharpe ratio corresponding to the primary asset trading strategy*

$$w(t) = k(t) \sigma_{diag}^{-1}(t) \rho^{-1}(t) s(t),$$

satisfies

$$s_X^2(t) = s'(t) \rho^{-1}(t) s(t).$$

Furthermore, the trading strategy

$$w(t) = k(t) \sigma_{diag}^{-1}(t) \rho^{-1}(t) \left(s(t) - \frac{s_\pi(t) - \rho'_\pi(t) \rho^{-1}(t) s(t)}{1 - \rho'_\pi(t) \rho^{-1}(t) \rho_\pi(t)} \rho_\pi(t) \right),$$

$$w_{N+1}(t) = k(t) \sigma_\pi^{-1}(t) \frac{s_\pi(t) - \rho'_\pi(t) \rho^{-1}(t) s(t)}{1 - \rho'_\pi(t) \rho^{-1}(t) \rho_\pi(t)},$$

taking positions in both the primary assets and the derivative has an instantaneous Sharpe ratio that satisfies

$$s_X^2(t) = s'(t) \rho^{-1}(t) s(t) + \frac{1}{1 - \rho'_\pi(t) \rho^{-1}(t) \rho_\pi(t)} (s_\pi(t) - \rho'_\pi(t) \rho^{-1}(t) s(t))^2 \geq s'(t) \rho^{-1}(t) s(t).$$

Finally, we note that

$$\sigma_\pi^2(t) (1 - \rho'_\pi(t) \rho^{-1}(t) \rho_\pi(t)) = \|\tilde{\Sigma}_\pi(t)\|^2, \quad \sigma_\pi(t) \rho'_\pi(t) \rho^{-1}(t) s(t) = \Sigma'_\pi(t) \theta^1(t).$$

Proof. The first result for the primary asset Kelly strategy is taken from [2]. For the second result, we define $\hat{w} = (w', w_{N+1})'$, $\hat{s} = (s', s_\pi)'$ and $\hat{\sigma}_{diag} = \text{diag}((\sigma', \sigma_\pi)')$, such that the extended Kelly strategy takes the form $\hat{w} = k \hat{\sigma}_{diag}^{-1} \hat{\rho}^{-1} \hat{s}$, with an associated instantaneous Sharpe ratio given by $s_X^2 = \hat{s}' \hat{\rho}^{-1} \hat{s}$. The expressions now follows from straightforward calculations and Lemma 5.1. Note that the last inequality follows from the bounds previously derived for the process h . For the final results, we first compare the elements of the inverse of the extended asset-asset covariance matrix

$$\hat{V}^{-1}(t) = \left(\begin{pmatrix} \Sigma(t) & \mathbf{0}_{N \times M-N} \\ \Sigma'_\pi(t) & \tilde{\Sigma}'_\pi(t) \end{pmatrix} \begin{pmatrix} \Sigma'(t) & \Sigma_\pi(t) \\ \mathbf{0}_{M-N \times N} & \tilde{\Sigma}_\pi(t) \end{pmatrix} \right)^{-1} = \begin{pmatrix} \Sigma(t) \Sigma'(t) & \Sigma(t) \Sigma_\pi(t) \\ (\Sigma(t) \Sigma_\pi(t))' & \sigma_\pi^2(t) \end{pmatrix}^{-1},$$

with those of the alternative expression $\hat{V}^{-1} = \hat{\sigma}_{diag}^{-1} \hat{\rho}^{-1} \hat{\sigma}_{diag}$. This shows that $\sigma_\pi^2 (1 - \rho'_\pi \rho^{-1} \rho_\pi) = \|\tilde{\Sigma}_\pi\|^2$, while the last identity is a direct consequence of Eq. (10). \square

By comparing Proposition 5.2 with Corollary 4.4 we see that we have found the trading strategies that attain the Hansen-Jagannathan bounds. It should come as no surprise that the bounds are attained by the Kelly strategy since this is an instantaneous Sharpe ratio maximal strategy. Proposition 5.2 also shows that a Kelly trader, applying a fixed Kelly multiplier k , is indifferent to trade the derivative if and only if

$$s_\pi(t) = \rho'_\pi(t) \rho^{-1}(t) s(t). \quad (32)$$

This leads us to one of the main results.

Theorem 5.3. *A Kelly trader is indifferent to trade the derivative if and only if every no-good-deal price $\{\Pi_c\}_{c \geq 0}$ is given by Π_0 , that is the unique minimal martingale measure no-arbitrage price.*

Proof. Without loss of generality we assume that $\|\tilde{\Sigma}_\pi\| > 0$ since otherwise the market collapses to a complete market. It is clear, from Definition 4.2, that if $\Pi_c = \Pi_0$, for all $c \geq 0$, then θ^2 is the zero-vector. Theorem 3.5 then states that $\sigma_\pi s_\pi = \Sigma'_\pi \theta^1$, which is the condition for the Kelly trader to be indifferent to trade the derivative. Hence, we are left with proving the other direction.

If the Kelly trader is indifferent to trade the derivative Theorem 3.5 states that the market price of risk satisfies

$$\theta^2(t) = \left(I - \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) \tilde{\Sigma}'_\pi(t) \right) \phi(t),$$

for some arbitrary \mathbb{F} -adapted process ϕ taking values in \mathbb{R}^{M-N} . As shown in the Appendix this implies that the static optimization problem, corresponding to the no-good-deal boundary prices for a fixed c , consists of finding the extremal solutions to

$$\Pi_c(t) \tilde{\Sigma}'_\pi(t) \theta^2(t), \quad \text{s.t.} \quad \|\theta^2(t)\| \leq c.$$

However, as $\tilde{\Sigma}'_\pi \theta^2$ is the zero-vector we make two conclusions. First, we notice that in this degenerate case

$$\hat{\theta}^2(t) = \pm \left(I - \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) \tilde{\Sigma}'_\pi(t) \right) \phi(t) \left\| \left(I - \|\tilde{\Sigma}_\pi(t)\|^{-2} \tilde{\Sigma}_\pi(t) \tilde{\Sigma}'_\pi(t) \right) \phi(t) \right\|^{-1} c.$$

Second, it follows from the result in the Appendix that the partial differential equations, associated with each market price of risk solution $\hat{\theta}^2$, are identical. Hence, given that the Kelly trader is indifferent to trade the derivative the no-good-deal price, for each fixed $c \geq 0$, is unique. Therefore, the price must equal Π_0 . \square

Corollary 5.4. *A Kelly trader is indifferent to trade the derivative if and only if every no-arbitrage price is given by the unique minimal martingale measure no-arbitrage price.*

Proof. The proof follows since the no-good-deal price, with index c , approaches the no-arbitrage price as $c \rightarrow \hat{c}$, according to Eq. (22). \square

So far we have shown that a Kelly trader can associate a unique no-arbitrage price to the derivative. This doesn't mean that the market must trade the derivative at this price. It only tells us that a Kelly trader can increase the magnitude of his instantaneous Sharpe ratio if the derivative is not traded at this price. We proceed by providing further evidence for why the market will price a derivative according to the Kelly indifference price given by the minimal martingale measure no-arbitrage price.

6 Kelly Equilibrium Pricing

If we consider the market as consisting of a number of investors we can expect their joint actions of trading to force the market to some kind of equilibrium. In this section we make precise what equilibrium means for a derivative product. We assume that each investor tries to maximize the magnitude of the instantaneous Sharpe ratio corresponding to his portfolio. This allows us to formulate the equilibrium as the solution to a reduced two-person zero sum differential game and apply the min-max concept of von Neumann [17]. It is well-known that any equilibrium found in this way is also a Nash equilibrium.

Let us consider a Kelly trader, applying an arbitrary Kelly multiplier k , who can invest in the primary assets and the derivative. If we denote the trading strategy by $\hat{w} = (w', w_{N+1})'$ then, as shown in Corollary 4.4 and Proposition 5.2, we have

$$\min_{\mu_\pi(t)} \max_{\hat{w}(t)} s_X^2(t) = s'(t) \rho^{-1}(t) s(t), \quad (33)$$

where the optimal trading strategy \hat{w}^* is defined as in Proposition 5.2 under the additional constraint that μ_π^* is set such that $s_\pi = \rho'_\pi \rho^{-1} s$. This choice of μ_π^* implies that the Kelly indifference price equals the minimal martingale measure no-arbitrage price and that the Kelly trader takes no positions in the derivative since $w_{N+1}^* = 0$.

Rather than looking at the interactions between every market investor we take the approach that all the other investors gang up on our Kelly trader. Hence, the other traders (not necessarily Kelly traders) will try to minimize the magnitude of the instantaneous Sharpe ratio for every trading strategy being used. The point of equilibrium can be described as below.

Theorem 6.1. *Given a Kelly trader who can invest in the primary assets and the derivative. Then*

$$\min_{\mu_\pi(t)} \max_{\hat{w}(t)} s_X^2(t) = \max_{\hat{w}(t)} \min_{\mu_\pi(t)} s_X^2(t) = s'(t) \rho^{-1}(t) s(t).$$

The equilibrium solution (μ_π^*, \hat{w}^*) is given by

$$\mu_\pi^*(t) = r(t) + \sigma_\pi(t) \rho'_\pi \rho^{-1}(t) s(t) - \frac{1}{2} \sigma_\pi^2, \quad w^*(t) = k(t) \sigma_{diag}^{-1}(t) \rho^{-1}(t) s(t), \quad w_{N+1}^*(t) = 0.$$

Proof. Since we have already shown the market coalition's response to a Kelly trader it remains to consider the response of a Kelly trader to the market coalition. The first order condition corresponding to the optimization problem $\min_{\mu_\pi} s_X^2$ is: $w_{N+1} = 0$. A Kelly trader facing such a constraint will trade in the primary assets using the strategy $w = k\sigma_{diag}^{-1}\rho^{-1}s$. Moreover, in order to enforce that $w_{N+1} = 0$ the market coalition will set μ_π such that $s_\pi = \rho'_\pi\rho^{-1}s$. This concludes the proof. \square

We have shown that the market is in equilibrium, in the sense described above, if and only if a Kelly trader is indifferent to trade the derivative. This is a very natural condition since any trader can either choose to be long or short the derivative and for each long/short position there is an opposite short/long counterpart. Hence, a derivative price resulting in a Kelly trader being indifferent to trade the derivative can indeed be seen as a fair price. Note also that if all the market participants are Kelly traders the market price must be in equilibrium since each contract requires a buyer and a seller. By the use of Theorem 5.3 we now state that the market is in equilibrium if and only if every no-good-deal price is given by the unique minimal martingale measure no-arbitrage price and the statement extends to the entire arbitrage-free price interval according to Corollary 5.4. This takes us back to the original paper of Black and Scholes [4] who used equilibrium arguments to derive the price of a derivative in a complete market setting. In this sense we generalize their approach to incomplete markets.

It is illustrative to analyze the situation where the market does not value the derivative according to the equilibrium price Π_0 . According to Eq. (8) and Theorem 6.1 we then have

$$d \log \frac{\Pi(t)}{\Pi_0(t)} = \sigma_\pi(t) (s_\pi(t) - s_\pi^*(t)) dt, \quad s_\pi^*(t) = \rho'_\pi(t) \rho^{-1}(t) s(t), \quad (34)$$

since the volatility of the derivative is invariant for all no-arbitrage prices. We therefore have

$$\log \frac{\Pi(t)}{\Pi_0(t)} = - \int_t^T \sigma_\pi(u) (s_\pi(u) - s_\pi^*(u)) du, \quad 0 \leq t \leq T. \quad (35)$$

Heuristically, we now draw the conclusion that $s_\pi \geq s_\pi^*$ if and only if $\Pi \leq \Pi_0$, continuously in time, and claim that a proper proof can be formulated by using the results presented in the Appendix. It then follows from Proposition 5.2 that $w_{N+1} \geq 0$ if and only if $\Pi \leq \Pi_0$, continuously in time, for every Kelly trader. If we further assume, as in [2], that buying (selling) an asset has a positive (negative) impact on the price of the asset, it is clear that every Kelly trader will push the market price of the derivative towards the equilibrium price should they differ at any point in time. Hence, without diving into the technical details we argue that the equilibrium price is dynamically stable if the Kelly traders dominate the market.

Let us end this section by describing how a Kelly trader goes about assessing a price to a derivative. For the sake of simplicity we focus on an optimal Kelly trader employing a Kelly multiplier $k = 1$. If the optimal Kelly trader is only allowed to trade in the N primary assets we know from Proposition 5.2 that the optimal trading strategy equals

$$w(t) = \sigma_{diag}^{-1}(t) \rho^{-1}(t) s(t). \quad (36)$$

The Kelly trader then adds $M - N$ derivatives to his opportunity set

$$d \log \Pi_m(t) = \mu_{\pi|m}(t) dt + \Sigma'_{\pi|m}(t) dW^1(t) + \tilde{\Sigma}'_{\pi|m}(t) dW^2(t), \quad 1 \leq m \leq M - N, \quad (37)$$

in such a way that the extended correlation matrix

$$\hat{\rho}(t) = \begin{pmatrix} \rho(t) & \rho_\pi(t) \\ \rho'_\pi(t) & \tilde{\rho}_\pi(t) \end{pmatrix}, \quad (38)$$

is invertible. Here, the blocks are defined, for each $1 \leq n \leq N$, according to

$$\rho_{\pi|n,m}(t) = \frac{\Sigma'_{\pi|m}(t) \Sigma_n(t)}{\sigma_{\pi|m}(t) \sigma_n(t)}, \quad \sigma_{\pi|m}^2(t) = \|\Sigma_{\pi|m}(t)\|^2 + \|\tilde{\Sigma}_{\pi|m}(t)\|^2, \quad 1 \leq m \leq M - N, \quad (39)$$

and

$$\tilde{\rho}_{\pi|u,v}(t) = \frac{\Sigma'_{\pi|u}(t) \Sigma_{\pi|v}(t) + \tilde{\Sigma}'_{\pi|u}(t) \tilde{\Sigma}_{\pi|v}(t)}{\sigma_{\pi|u}(t) \sigma_{\pi|v}(t)}, \quad 1 \leq u, v \leq M - N. \quad (40)$$

If we further let $\hat{\sigma}_{diag} = \text{diag} \left((\sigma', \sigma_{\pi|1}, \dots, \sigma_{\pi|M-N})' \right)$, elementary matrix manipulations yield

$$\begin{pmatrix} w(t) \\ \mathbf{0}_{M-N} \end{pmatrix} = \hat{\sigma}_{diag}^{-1}(t) \begin{pmatrix} \rho^{-1}(t) s(t) \\ \mathbf{0}_{M-N} \end{pmatrix} = \hat{\sigma}_{diag}^{-1}(t) \hat{\rho}^{-1}(t) \begin{pmatrix} s(t) \\ \rho'_\pi(t) \rho^{-1}(t) s(t) \end{pmatrix}. \quad (41)$$

We now consider an optimal Kelly strategy which takes positions in both the primary assets and in the derivatives. Such a strategy is given by $\hat{w} = \hat{\sigma}_{diag}^{-1} \hat{\rho}^{-1} \hat{s}$, where $\hat{s} = (s', s'_\pi)'$ and where $s_\pi = (s_{\pi|1}, \dots, s_{\pi|M-N})'$ denotes the instantaneous Sharpe ratio of the derivatives. A simple comparison with Eq. (41) then shows that the indifference price corresponding to each of the added derivatives satisfies the vector equation

$$s_\pi(t) = \rho'_\pi(t) \rho^{-1}(t) s(t). \quad (42)$$

Since the trading strategy of any Kelly trader is proportional to that of an optimal Kelly trader (via the Kelly multiplier) it follows that every Kelly trader agrees on the indifference prices. Note further that the condition related to the indifference price, Eq. (42), is independent on whether the market is complete or incomplete. Should the price of a derivative not equal the indifference price a Kelly trader will add the derivative to his portfolio and thereby increasing his instantaneous Sharpe ratio. The contribution of this paper is to identify the Kelly indifference price with the unique minimal martingale no-arbitrage price as characterized in Definition 3.8 with $\theta^1 = \Sigma^{-1} \sigma_{diag} s$ and θ^2 being equal to the zero-vector. This shows that under the additional assumption of a complete market, in which case the matrix $\tilde{\Gamma}$ vanishes, the Kelly indifference price agrees with the unique no-arbitrage price of Merton [14].

7 Conclusions

In this paper we show that if the capital market is in equilibrium, in the sense that a Kelly trader can do no better (in terms of maximizing the magnitude of his instantaneous Sharpe ratio) by trading the derivative given the actions of the other market participants, the no-arbitrage price of the derivative must equal that of the minimal martingale measure no-arbitrage price in [7]. The proof relies on showing that a Kelly trader is indifferent to trade the derivative if and only if every no-good-deal price in [5] is unique. Since the maximal no-good-deal price interval corresponds to the no-arbitrage price interval and a Kelly trader is indifferent to trade the derivative when the market is in equilibrium the result follows. Hence, by adding elements from the optimal growth theory of Kelly [12] and Latané [13] we derive a unique indifference and equilibrium price for derivatives in incomplete markets.

A Appendix

In this Appendix we state the Hamilton-Jacobi-Bellman equation associated with the upper and lower no-good-deal prices. Without loss of generality we can view the non-tradable indices as factors such that the model parameters allows a Markovian representation with respect to (P, I) . This means, for instance, that there exists a function $r : \mathbb{R}_+ \times \mathbb{R}_+^N \times \mathbb{R}^{M-N} \rightarrow \mathbb{R}$ such that

$$r(t) = r(t, P(t), I(t)),$$

and similar for the other terms. Additionally, we assume that the market price of risk θ also allows for a Markovian representation. Let further $F : \mathbb{R}_+ \times \mathbb{R}_+^N \times \mathbb{R}^{M-N} \rightarrow \mathbb{R}_+$ be a deterministic function, such that the infinitesimal operator \mathcal{A}^θ of (P, I) , with respect to the probability measure \mathbb{P}_θ , takes the form

$$\begin{aligned} \mathcal{A}^\theta F &= \sum_{i=1}^N \frac{\partial F}{\partial x_i} x_i r(t, x, y) + \sum_{j=1}^{M-N} \frac{\partial F}{\partial y_j} \left(\gamma_j(t, x, y) - \Gamma'_j(t, x, y) \theta^1(t, x, y) - \tilde{\Gamma}'_j(t, x, y) \theta^2(t, x, y) \right) \\ &+ \frac{1}{2} \sum_{i,k=1}^N \frac{\partial^2 F}{\partial x_i \partial x_k} x_i x_k \Sigma'_i(t, x, y) \Sigma_k(t, x, y) + \sum_{i=1}^N \sum_{j=1}^{M-N} \frac{\partial^2 F}{\partial x_i \partial y_j} x_i \Sigma'_i(t, x, y) \Gamma_j(t, x, y) \\ &+ \frac{1}{2} \sum_{j,k=1}^{M-N} \frac{\partial^2 F}{\partial y_j \partial y_k} \left(\Gamma'_j(t, x, y) \Gamma_k(t, x, y) + \tilde{\Gamma}'_j(t, x, y) \tilde{\Gamma}_k(t, x, y) \right). \end{aligned}$$

We now specify the function

$$F_c(t, x, y) = B(t) \mathbb{E}_\theta \left[\frac{\Phi(P(T))}{B(T)} \mid P(t) = x, I(t) = y \right], \quad \theta \in \Theta_T^M, \quad c \in \mathbb{R}_+,$$

for some $\Phi : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$, such that

$$\Pi_c(t) = F_c(t, P(t), I(t)), \quad a.s. \quad 0 \leq t \leq T.$$

It is proven in [3] that the no-good-deal price boundaries ($\min \Pi_c, \max \Pi_c$) can be characterized by two deterministic functions (F_c^1, F_c^2). One of these functions solve the Hamilton-Jacobi-Bellman equations:

$$\frac{\partial F_c}{\partial t}(t, x, y) + \sup_{\|\theta^2\| \leq c} \mathcal{A}^\theta F_c(t, x, y) - r(t, x, y) F_c(t, x, y) = 0,$$

$$F_c(T, x, y) = \Phi(x),$$

while the other solves a similar equation but with sup exchanged for inf. Hence, deriving the no-good-deal boundary prices can be composed in two steps. First, compute the extremal points of

$$\sum_{j=1}^{M-N} \frac{\partial F_c}{\partial y_j}(t, x, y) \tilde{\Gamma}'_j(t, x, y) \theta^2(t, x, y), \quad s.t. \quad \|\theta^2(t, x, y)\| \leq c.$$

Thereafter, solve the partial differential equation for each θ^2 . By introducing the gradient $\nabla_y F_c = \left(\frac{\partial F_c}{\partial y_1}, \dots, \frac{\partial F_c}{\partial y_{M-N}} \right)'$, it is easily seen that the static optimization problem takes the simple form

$$\hat{\theta}^2(t, x, y) = \pm \|\tilde{\Gamma}'(t, x, y) \nabla_y F_c(t, x, y)\|^{-1} \tilde{\Gamma}'(t, x, y) \nabla_y F_c(t, x, y) c.$$

We further notice that every no-arbitrage price (and hence every no-good-deal price) satisfies the same partial differential equation, albeit with an arbitrary market price of risk component θ^2 . This implies that

$$\tilde{\Sigma}_\pi(t, x, y) = \frac{1}{F_c(t, x, y)} \tilde{\Gamma}'(t, x, y) \nabla_y F_c(t, x, y), \quad \forall c \geq 0.$$

Hence, the static optimization problem can equally be stated as finding the extremal points of

$$F_c(t, x, y) \tilde{\Sigma}'_\pi(t, x, y) \theta^2(t, x, y), \quad s.t. \quad \|\theta^2(t, x, y)\| \leq c.$$

For completeness, we further note that also $\frac{1}{F_c} \nabla_x F_c$ is independent of c , where $\nabla_x F_c = \left(\frac{\partial F_c}{\partial x_1}, \dots, \frac{\partial F_c}{\partial x_N} \right)'$.

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Kelly trading and option pricing

HANS-PETER BERMIN & MAGNUS HOLM

In this paper we show that a Kelly trader is indifferent to trade the derivative if and only if the no-arbitrage price is uniquely given by the minimal martingale measure no-arbitrage price, thus providing a natural selection mechanism for option pricing in incomplete markets. We also show that the unique Kelly indifference price results in market equilibrium, in the sense that no Kelly trader can improve the magnitude of his instantaneous Sharpe ratio, by trading the derivative, given the actions of the other market participants.

KEYWORDS: Option pricing, Incomplete markets, Hansen-Jagannathan bound, Minimal martingale measure, Kelly Indifference price, Kelly Equilibrium

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